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ACCELERATION OF SERIES

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ABSTRACT

The rate of convergence of infinite series can be accelerated by a suitable splitting of each term into two parts and then combining the second part of the n-th term with the first part of the (n+1)-th term to get a new series and leaving the first part of the first term as an "orphan". Repeating this process an infinite number of times, the series will often approach zero, and we obtain the series of orphans, which may converge faster than the original series. Heuristics for determining the splits are given. Various mathematical constants, originally defined as series having a term ratio which approaches 1, are accelerated into series having a term ratio less than 1. This is done with the constants of Euler and Catalan. The series for pi/4 = arctan 1 is transformed into a variety of series, among which is one having a term ratio of 1/27 and another having a term ratio of 54/3125. A series for 1/pi is found which has a term ratio of 1/64 and each term of which is an integer divided by a power of 2, thus making it easy to evaluate the sum in binary arithmetic. We express zeta(3) in terms of pi↑3 and a series having a term ratio of 1/16. Various hypergeometric function identities are found, as well as a series for (arcsin y)↑2 curiously related to a series for y arcsin y. Convergence can also be accelerated for finite sums, as is shown for the harmonic numbers. The sum of the reciprocals of the Fibonacci numbers has been expressed as a series having the convergence rate of a theta function. Finally, it is shown that a series whose n-th term ratio is (n+p)(n+q)/(n+r)(n+s), where p, q, r, s are integers, is equal to c + d pi↑2, where c and d are rational.

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Acceleration of Series

Dedicated to the memory of Srinivasa Ramanujan Aiyangar,

Indian Summer,

1973.

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Introduction

This paper describes a way to manipulate sums to produce new ones which converge faster. For example, knowing only that

$$pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots$$

we can find that

$$\frac{pi}{4} = \frac{2}{15} \left(6 - \frac{1}{3} \frac{1}{7} \frac{2}{11} \left(13 - \frac{1}{3} \frac{3}{13} \frac{4}{17} \left(20 - \frac{1}{3} \frac{5}{19} \frac{6}{23} \left(27 - \dots\right.\right.\right.\right.$$

without even using calculus. Now the first series is a lot simpler, but it would require billions of terms to get 10 digits of pi, whereas the second series only needs 7 terms, since each term is less than 1/27th of the previous one.

Another example is the sum of the reciprocals of the cubes:

$$\zeta 3 = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots = \sum_{n\geq 1} \frac{1}{n^3}.$$

We will show that this slow series

$$= \frac{5}{4}\left(\frac{1}{1} - \frac{1}{3}\frac{1}{4}\left(\frac{1}{2} - \frac{2}{5}\frac{2}{6}\left(\frac{1}{3} - \frac{3}{7}\frac{3}{8}\left(\dots\right.\right.\right.\right.$$

$$= \frac{5}{2}\sum_{k\geq 1} \frac{(-)^{k-1}}{k^3 \binom{2k}{k}},$$

where each term is less than 1/4 of the previous.

The treatment will be informal with few, if any, proofs.

The method consists of splitting each series term into two pieces, then recombining the pieces of adjacent terms to form a new series with smaller terms, and a leftover quantity called an "orphan". This process is repeated until a formula for the orphans is deduced, wherupon the shrinking series is discarded in favor of the series of orphans. The simplest case of this is an old trick known as

Euler's Transform

Although it is usually applied to alternating series, Euler's transform is more neatly derived for the general series

$$\sum_{n\geq 0} a_n = a_0 + a_1 + a_2 + a_3 + \dots$$

$$= \frac{a_0}{2} + \frac{a_0 + a_1}{2} + \frac{a_1 + a_2}{2} + \frac{a_2 + a_3}{2} + \dots$$

$$= \frac{a_0}{2} + \sum_{n\geq 0} U a_n$$

where U is the averaging operator, $U a_n = \dfrac{a_n + a_{n+1}}{2}$.

Thus we get the orphan term $(a_0/2)$ and a new series upon which

we may repeat this operation to get

$$= \frac{a_0}{2} + \frac{U a_0}{2} + \sum_{n \ge 0} U^2 a_n$$

where $U^2 a_n = U (U a_n) = \dfrac{a_n + 2 a_{n+1} + a_{n+2}}{4}$.

If we do this k times we will get

$$= \frac{a_0}{2} + \frac{U a_0}{2} + \dots + \frac{U^{k-1} a_0}{2} + \sum_{n \ge 0} U^k a_n .$$

Now as k approaches infinity, the sum on the right approaches 0 for all convergent (and many divergent) series, leaving, finally,

$$\sum_{n \ge 0} a_n = \frac{1}{2} \sum_{k \ge 0} U^k a_0$$

$(U^0 a_n = a_n)$. The motivation for this transform should

become clear with an example. We have

pi/4 = arctan 1 = 1 - 1/3 + 1/5 - 1/7 + ...

so that $a_n = \dfrac{(-)^n}{2n+1}$

$$U a_n = \frac{1}{2}\left(\frac{(-)^n}{2n+1} + \frac{(-)^{n+1}}{2n+3}\right) = \frac{(-)^n}{(2n+1)(2n+3)}$$

$$U^2 a_n = \frac{1}{2}\left(\frac{(-)^n}{(2n+1)(2n+3)} + \frac{(-)^{n+1}}{(2n+3)(2n+5)}\right) = \frac{2\ (-)^n}{(2n+1)(2n+3)(2n+5)}$$

It is easy to induce that

$$U^k a_n = \frac{k!\ (-)^n}{(2n+1)\ \ldots\ (2n+2k+1)} .$$

Setting n to 0, we find

$$\frac{pi}{4} = \sum_{n\geq 0} \frac{(-)^n}{2n+1} = \frac{1}{2}\sum_{k\geq 0} \frac{1}{3}\,\frac{2}{5}\,\frac{3}{7} \cdots \frac{k}{2k+1} .$$

The left hand sum has as a ratio of consecutive terms

$$-\frac{2n+1}{2n+3}$$, which approaches -1 as n grows large.

To get 9 digits of pi would require about half a billion terms. The right hand sum is easier to compute than it looks, since

its kth term is just $\frac{k}{2k+1}$ times the previous.

Since this term ratio approaches 1/2 as k grows, only 30 terms are needed for 9 digits (30 bits). This convergence rate of 1 bit/term is typical of Euler transformed series.

Note, however, that if by ugly coincidence some different series happened to match our pi/4 series for the first 30 terms (and then, for instance, became all zeros), the first 30 orphans would have the same numerical values, albeit from a different formula, in which case few, if any, of the 9 digits would be right.
Moral: virtually nothing is certain until you have the formula for the kth orphan.

Generalization of the Euler Transform

Unfortunately, if the original series converges more rapidly than 1 bit/term (term ratio < 1/2), the transformed series may converge less rapidly, as we shall see. This is a symptom of the fact that Euler's transform is most effective when consecutive terms are of nearly equal magnitude and opposite sign, so that their averages will be very small, leaving most of the sum's value in the orphan (More orphan than not). What we need is an unsymmetrical, weighted averaging operator to achieve this near telescopy in series where each term is significantly smaller than its predecessor. We are indebted to Rich Schroeppel for suggesting the weights 1/(1-r) and r/(r-1) be used when the series has a limiting term ratio r. This will actually split the terms of non-alternating series (positive r) into pieces with opposite sign. It was an easy step from this to realize that the weights need not be the same for each term.

Thus we will denote a pair of weights by s_n and $1 - s_n$,

and call s_n a splitting function.

We generalize Eulsr's transform as follows:

$$\sum_{n\geq 0} a_n = \sum_{n\geq 0} (s_n + 1 - s_n)\, a_n$$

$$= s_0 a_0 + \sum_{n\geq 0} (1 - s_n)\, a_n + s_{n+1} a_{n+1}$$

which specializes to the first step of Euler's transform when $s_n = 1/2$. $s_n = 0$ leaves the series alone, while $s_n = 1$

translates the series by one term, the zeroth becoming the orphan. We wish to choose s_n, the splitting function, so

as to make the summand both small and simple; small so that upon repeated applications of this operation, the value of the sums will accrue rapidly in the orphans, and simple so that these orphans have a general formula easy to discover and compute.

For examples, let $s_n = s$, a constant. Transforming the geometric series:

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n = s + \sum_{n\geq 0} (1-s)\, x^n + s\, x^{n+1}$$

$$= s + (1 - s + s\,x) \sum_{n\geq 0} x^n$$

$$= s + (1 - s + s\,x)\, s + (1 - s + s\,x)^2\, s + \ldots$$

$$= s \sum_{k\geq 0} (1 - s + s\,x)^k = \frac{s}{1 - (1 - s + s\,x)}$$

an identity, except the interval of convergence has moved from $-1 < x < 1$ to $1 - 2/s < x < 1$ or $1 < x < 1 - 2/s$, depending on the sign of s.

If we choose s so that $1 - s + s\,x = 0$, all of the terms of this last sum vanish except the first, leaving s which is indeed $1/(1-x)$. We note that if $s = 1/2$ (Euler's transform) and $-1/3 < x < 1/2$, the transformed series converges more slowly than the original geometric series.

Integrating these series from 0 to x:

$$- \ln\,(1 - x) = \sum_{n\geq 1} \frac{x^n}{n} = \sum_{k\geq 1} \frac{(1 - s + s\,x)^k}{k} - \frac{(1 - s)^k}{k}$$

$$= \ln\,(s - s\,x) - \ln s$$

Here, if we again choose $s = 1/(1-x)$

$$\sum_{n\geq 1} \frac{x^n}{n} = -\sum_{k\geq 1} \frac{\left(\frac{x}{x-1}\right)^k}{k}$$

or $$-\ln(1-x) = \ln\left(1 - \frac{x}{x-1}\right)$$

If $x = -1/2$, we can compute $\ln(3/2)$ with an asymptotic term ratio of 1/3 using the transformed series, versus -1/2 for the original.

If we had chosen $s = 2/(2-x)$ instead of $1/(1-x)$, then $1 - s + s\,x = -(1-s)$, that is, alternate terms in the sum for $\ln(s - s\,x) - \ln s$ will cancel, leaving

$$\sum_{n\geq 1} \frac{x^n}{n} = 2\sum_{k\geq 0} \frac{\left(\frac{x}{2-x}\right)^{2k+1}}{2k+1}$$

or $$-\ln(1-x) = \ln\frac{1+y}{1-y} \quad \text{with} \quad y = \frac{x}{2-x}$$

Here, if $x = -1$, the term ratio approaches 1/9, yielding $\ln 2$ with almost 1 decimal digit/term instead of 1 bit/term, as with Euler's transform.

The formulas we have seen up to now have been known for a long time. Before we get into the more interesting goodies, I want to replace the summation notation with one that stresses term ratios, and reformulate the series transform accordingly.

R Notation

Let $R_{n\geq m}\, r_n$ stand for the series whose first term is 1, second term is r_m and whose ratio of n+1st to nth term is r_n,

i.e., $1 + r_m (1 + r_{m+1} (1 + \ldots$.

Thus

$$a_m + a_{m+1} + \ldots = a_m \, R_{n\geq m} \frac{a_{n+1}}{a_n} .$$

Note that m need not be an integer:

$$\frac{pi}{4} = \underset{n\geq 0}{R} - \frac{2n+1}{2n+3} = \underset{n\geq \frac{1}{2}}{R} - \frac{n}{n+1} .$$

Other examples of R notation:

$$e^{x} = \sum_{n\geq 0} \frac{x^{n}}{n!} = \underset{n\geq 1}{R} \frac{x}{n}$$

$$\ln(1+x) = 2 \sum_{k\geq 0} \frac{\left(\frac{x}{2+x}\right)^{2k+1}}{2k+1}$$

$$= \frac{2x}{x+2} \underset{k\geq 0}{R} \frac{2k+1}{2k+3} \left(\frac{x}{x+2}\right)^{2}$$

$$F(a+1,b+1;c+1;z) = \frac{c!}{a!\,b!} \sum_{n\geq 0} \frac{(n+a)!\,(n+b)!}{(n+c)!\,n!} z^{n}$$

$$= \underset{n\geq 1}{R} \frac{n+a}{n+c} \frac{n+b}{n} z$$

(F is the hypergeometric function. Unfortunately, a, b, and c are offset by 1, apparently to accomodate the Gamma notation for the factorial function. The author finds this unnatural and pedantic.)

As it stands, this notation cannot express a series starting with 0. Also, the (2 k + 1)/(2 k + 3) in the logarithm formula is awkward--suggesting the following generalization:

Let $b_n \underset{n\geq 0}{R} r_n$ stand for the series

$$b_0 + r_0 (b_1 + r_1 (b_2 + \ldots =$$

$$b_0 + r_0 b_1 + r_0 r_1 b_2 + r_0 r_1 r_2 b_3 + \ldots$$

i.e. $b_0 \mathop{R}\limits_{n \ge 0} \frac{b_{n+1}}{b_n} r_n$. Another way to say it is that

$$b_n \mathop{R}\limits_{n \ge m} r_n = S_m$$

where S satisfies the first order, linear recurrence relation

$$S_m = b_m + r_m S_{m+1} .$$

Then our logarithm formula becomes

$$\ln (1 + x) = \frac{2 x}{x + 2} (2 k + 1) \mathop{R}\limits_{k \ge 0} \left(\frac{x}{x + 2}\right)^2 .$$

Now our notation is no longer uniquely determined by the series--in fact one could cop out and write

$$a_m + a_{m+1} + \ldots = a_n \mathop{R}\limits_{n \ge m} 1 .$$

The intended convention, however, is that only factors of the nth, but not the n+1st term appear to the left of the R. (And, of course, the necessary scale factor to make the first term come out right.) This will prevent factors of the form (k + a)/(k + a + 1) or its reciprocal from needlessly complicating the expression to the right of the R, whose primary purpose is to get rid of nth powers and factorials.

Thus, Ramanujan's amazing pi formula reads

$$\frac{4}{pi} = \frac{21460 n + 1123}{882} \mathop{R}\limits_{n \ge 0} - \frac{(4 n + 1) (2 n + 1) (4 n + 3)}{128\ 21^4 (n + 1)^3}$$

(Collected Papers, #6, eq 39).

The advantages of this notation are:

One can tell at a glance the first term and term ratio, which immediately indicates the required computation effort and convergence rate. The number of digits/term is just $-\log \lim_n |r_n|$ as n grow large, where the base of the log is the radix of the digits.

The notation is usually more concise, since term ratios are usually simpler than terms. This often makes it easier to notice when a series is equal to, or a special case of, another.

It suggests a generally more efficient method of numerically evaluating n terms of a sum-- instead of computing the terms and adding them up, take r_n, add b_n, multiply by r_{n-1} add b_{n-1}, ... multiply by r_0, and finally add b_0.

It has suggested two "spigot" methods of evaluating sums to unlimited accuracy without multiplying or dividing very long numbers. A "spigot method" is one that produces digits on demand, computing only as much as necessary, yet able to supply subsequent digits upon subsequent demand, without, in effect, starting over.

It is immediately rewritable as a continued fraction:

$$b_n \mathop{\mathrm{R}}_{n\geq 0} \frac{p_n}{q_n} = b_0 + \cfrac{p_0}{0 + \cfrac{q_0}{b_1 + \cfrac{p_1}{0 + \cfrac{q_1}{\ldots}}}}$$

Although it isn't quite as manipulable as summation notation, we can do things like

$$\frac{d}{dx}\, x^a \mathop{\mathrm{R}}_{k\geq 0} r_k\, x^b = (k\,b + a)\, x^{a-1} \mathop{\mathrm{R}}_{k\geq 0} r_k\, x^b$$

Finally, it simplifies the forthcoming discussion of series transformations.

Derivation of the Transform

After applying k-1 splitting functions, thereby extracting k-1 orphans, we are ready to apply the kth splitting function $s_{k,n}$ to the remaining series, whose nth term we denote $a_{k,n}$, so that $a_{0,n} = a_n$, the nth term of the original series. Also,

$$r_{k,n} = \frac{a_{k,n+1}}{a_{k,n}}$$

the nth term ratio of the ktuply accelerated series. Then,

$$\sum_{n\geq m} a_{k,n} = \sum_{n\geq m} a_{k,n} (s_{k,n} + 1 - s_{k,n})$$

$$= s_{k,m} a_{k,m} + \sum_{n\geq m} (1 - s_{k,n}) a_{k,n} + s_{k,n+1} a_{k,n+1}$$

$$= s_{k,m} a_{k,m} + \sum_{n\geq m} (1 - s_{k,n} + s_{k,n+1} r_{k,n}) a_{k,n}$$

and now if we define $u_{k,n} = 1 - s_{k,n} + r_{k,n} s_{k,n+1}$

$$\sum_{n\geq m} a_{k,n} = s_{k,m} a_{k,m} + \sum_{n\geq m} u_{k,n} a_{k,n}$$

so $s_{k,m} a_{k,m}$ is the kth orphan, and the nth term of our new series is $u_{k,n} a_{k,n}$, which we define to be $a_{k+1,n}$. Thus

$$\sum_{n\geq m} a_{k,n} = s_{k,m} a_{k,m} + \sum_{n\geq m} a_{k+1,n}$$

As before, we iterate on k indefinitely, choosing $s_{k,n}$ so that the $u_{k,n}$ and thus the $a_{k,n}$ and thus their sums tend to 0 with k as rapidly as possible. Then nothing will remain of the original sum but the aggregate of the orphans. If k is initially j,

$$\sum_{n\geq m} a_{j,n} = s_{j,m}\, a_{j,m} + s_{j+1,m}\, a_{j+1,m} + \ldots$$

$$= s_{j,m}\, a_{j,m} + s_{j+1,m}\, u_{j,m}\, a_{j,m} + s_{j+2,m}\, u_{j+1,m}\, u_{j,m}\, a_{j,m} + \ldots$$

because $a_{k+1,n} = u_{k,n}\, a_{k,n}$ by definition.

In R notation, this last equation reduces to

$$a_{j,m} \underset{n\geq m}{R}\, r_{j,n} = a_{j,m}\, s_{k,m} \underset{k\geq j}{R}\, u_{k,m} .$$

Note that we have nowhere used the fact that j is initially 0, or even an integer, so that we are free to choose any convenient value to denote our original series.

We now repeat this whole derivation in R notation, skipping fewer steps. The reader is advised to understand it, since the main results in this paper depend heavily on it.

For convenience, we assume that the original series has been scaled so that $a_{j,m} = 1$. Then

$$\sum_{n\geq m} a_{j,n} = \underset{n\geq m}{R}\, r_{j,n} = (1 - s_{j,n} + s_{j,n}) \underset{n\geq m}{R}\, r_{j,n}$$

$$= (1 - s_{j,n}) \underset{n\geq m}{R}\, r_{j,n} + s_{j,n} \underset{n\geq m}{R}\, r_{j,n}$$

$$= (1 - s_{j,n}) \underset{n\geq m}{R}\, r_{j,n} + s_{j,m} + r_{j,m}\, s_{j,n+1} \underset{n\geq m}{R}\, r_{j,n+1}$$

$$= (1 - s_{j,n}) \mathrm{R}_{n \geq m} r_{j,n} + s_{j,m} + r_{j,m} s_{j,n+1} \mathrm{R}_{n \geq m} r_{j,n+1} \frac{r_{j,n}}{r_{j,n}}$$

$$= (1 - s_{j,n}) \mathrm{R}_{n \geq m} r_{j,n} + s_{j,m} + r_{j,m} \frac{r_{j,n}}{r_{j,m}} s_{j,n+1} \mathrm{R}_{n \geq m} r_{j,n}$$

$$= (1 - s_{j,n}) \mathrm{R}_{n \geq m} r_{j,n} + s_{j,m} + r_{j,n} s_{j,n+1} \mathrm{R}_{n \geq m} r_{j,n}$$

$$= s_{j,m} + (1 - s_{j,n} + r_{j,n} s_{j,n+1}) \mathrm{R}_{n \geq m} r_{j,n}$$

$$= s_{j,m} + u_{j,n} \mathrm{R}_{n \geq m} r_{j,n}$$

$$= s_{j,m} + u_{j,m} \mathrm{R}_{n \geq m} \frac{u_{j,n+1}}{u_{j,n}} r_{j,n}$$

$$= s_{j,m} + u_{j,m} \mathrm{R}_{n \geq m} r_{j+1,n}$$

$$\left(r_{k+1,n} = \frac{u_{k,n+1}}{u_{k,n}} r_{k,n} \text{ because } a_{k+1,n} = u_{k,n} a_{k,n}\right)$$

$$= s_{j,m} + u_{j,m} (s_{j+1,m} + u_{j+1,m} \mathrm{R}_{n \geq m} r_{j,n+2})$$

$$= s_{j,m} + u_{j,m} (s_{j+1,m} + u_{j+1,m} (s_{j+2,m} + \ldots$$

$$= s_{k,m} \mathrm{R}_{k \geq j} u_{k,m} .$$

Finally, we have the

Series Acceleration Formula

$$\underset{n\geq m}{R}\ r_{j,n} = s_{k,m}\ \underset{k\geq j}{R}\ u_{k,m}$$

Our problem then boils down to finding a pair of functions $s_{k,n}$ and $r_{k,n}$ which satsify the recurrence

$$r_{k+1,n} = \frac{u_{k,n+1}}{u_{k,n}}\ r_{k,n}$$

with $u_{k,n} = 1 - s_{k,n} + r_{k,n}\ s_{k,n+1}$

while for eome j, $r_{j,n}$ equals the term ratio of the series

we wish to accelerate.

Constant Splitting Functions

Since log and arctan are particularly useful, we will honor them with yet another example of a new way to get an old result:

$$-\ln\,(1 - x) = x\ \underset{n\geq 1}{R}\ \frac{n}{n+1}\ x$$

$$\arctan x = x\ \underset{n\geq 0}{R}\ \frac{n + 1/2}{n + 3/2}\ (-x^2)$$

These are just special cases of the hypergeometric function with 2nd argument = 1,

$$F(a,1;c;z) = \underset{n\geq 0}{R}\ \frac{n + a}{n + c}\ z\ .$$

Here $m = 0$ and again, our asymptotic term ratio is z, so if we continue to restrict $s_{k,n}$ to be a constant, $1/(1-z)$ is still our best guess to minimize $u_{k,n}$. This follows from setting $s_{k,n}$ and $s_{k,n+1}$ to s, $r_{k,n}$ to z, and $u_{k,n}$ to 0 in the definition of $u_{k,n}$.

We choose $j = 0$ and have

$$r_{0,n} = \frac{n + a}{n + c} z$$

$$u_{0,n} = 1 - \frac{1}{1 - z} + \frac{1}{1 - z} \frac{n + a}{n + c} z$$

$$= \frac{c - a}{n + c} \frac{z}{z - 1}$$

so

$$r_{1,n} = \frac{n + a}{n + c + 1} z .$$

Since this is just $r_{0,n}$ with $c+1$ replacing c, and c is arbitrary, we can immediately conclude that

$$u_{k,n} = \frac{c + k - a}{n + c + k} \frac{z}{z - 1} .$$

Finally,

$$\underset{n\geq 0}{R} \frac{n + a}{n + c} z = \frac{1}{1 - z} \underset{k\geq 0}{R} \frac{k + c - a}{k + c} \frac{z}{z - 1} .$$

or more symmetrically,

$$-z \underset{n\ge a}{\mathrm{R}} \frac{n}{n+d}\, z = \frac{z}{z-1} \underset{n\ge d}{\mathrm{R}} \frac{n}{n+a}\, \frac{z}{z-1}$$

(with d for c-a), a result usually found by analytic methods (See Abramowitz and Stegun, Handbook of Mathematical Functions, Formula 15.3.5).

(Note that the same transform on this new series yields the old one again, so that for every z for which convergence was improved, there was the point z/(z-1) for which it was degraded, and vice versa.)

In particular, we have

$$\arctan x = \frac{x}{x^2+1} \underset{k\ge 0}{\mathrm{R}} \frac{k+1}{k+3/2}\, \frac{x^2}{x^2+1},$$

the interval of convergence having expanded from ± 1 to $\pm$ infinity. Setting $y^2 = x^2/(x^2+1)$ we get

$$\frac{\arcsin y}{(1-y^2)^{1/2}} = y \underset{k\ge 1}{\mathrm{R}} \frac{k}{k+1/2}\, y^2 = \sum_{k\ge 0} \frac{y^{2k+1}}{\binom{k+1/2}{k}}$$

where

$$\binom{j}{k} = \frac{j!}{k!\,(j-k)!} = \binom{j}{j-k}$$

which, for integer k

$$= \frac{j\,(j-1)\,\ldots\,(j-k+1)}{k!}$$

so that

$$\binom{n-1/2}{k} = \frac{\binom{2n}{2k}\binom{2k}{k}}{\binom{n}{k}\, 4^k}.$$

Integrating the arcsin equation:

$$\frac{\arcsin^2 y}{2} = \sum_{k \ge 0} \frac{y^{2k+2}}{(2k+2)\binom{k+1/2}{k}}.$$

$$= \frac{y^2}{2} \mathop{R}_{k \ge 1} \frac{k^2}{(k+1/2)(k+1)} y^2$$

$$= \frac{y^2}{2k} \mathop{R}_{k \ge 1} \frac{k}{k+1/2} y^2$$

This result, although known, is peculiar in the light of

$$\arcsin y = \sum_{k \ge 0} \frac{\binom{k-1/2}{k} y^{2k+1}}{2k+1}$$

$$= \frac{y}{2k+1} \mathop{R}_{k \ge 0} \frac{k+1/2}{k+1} y^2$$

wherein the coefficients are nearly the reciprocals of those in the previous sum. The relationship is even more striking if we write

$$\arcsin y = \frac{y}{2k} \mathop{R}_{k \ge \frac{1}{2}} \frac{k}{k+1/2} y^2$$

which says that if we offset the index of summation by $-1/2$ in the formula for $\arcsin^2 y$, we get $y \arcsin y$.

Other Splitting Functions

Actually, for all but the simplest term ratios, most splitting functions, constant or otherwise, will lead to prohibitively complicated $u_{k,n}$'s after just a few k.

So the main problem is to find an $s_{k,n}$ such that $u_{k,m}$ stays simple. The easiest way to do this seems to be to start with $r_{0,n}$ and choose the $s_{0,n}$ which results in the simplest $r_{1,n}$ you can find. Then repeat the procedure for $r_{1,n}$, seeking the $s_{1,n}$ leading to the simplest possible $r_{2,n}$. With some skill and luck, you find a sequence $s_{0,n}$, $s_{1,n}$, ... which produces a sequence of u's whose general formula is discernable.

For example, let us attack

$$\text{zeta } 2 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = R_{n\geq 1} \frac{n^2}{(n+1)^2} .$$

It will prove convenient to choose j = 1 in the Acceleration Formula, and so define

$$r_{1,n} = \frac{n^2}{(n+1)^2} .$$

We certainly can't try $s_{1,n} = \text{Lim}_n\, 1/(1-r_n)$, since the asymptotic term ratio is 1, so let us try a very simple function of n

$$s_{1,n} = a\,n + b$$

with a and b to be determined so as to simplify

$$u_{1,n} = \frac{(1-a)\,n^2 + (2-a-2b)\,n + 1-b}{(n+1)^2}.$$

A good heuristic is to reduce the degree of u, which we can do twice by choosing a = 1, b = 1/2, so that

$$u_{1,n} = \frac{1}{2\,(n+1)^2}.$$

Then

$$r_{2,n} = \frac{n^2}{(n+2)^2}.$$

Trying the same form again,

$$s_{2,n} = a\,n + b,$$

we have

$$u_{2,n} = \frac{(1-3a)\,n^2 + (4-4a-4b)\,n + 4-4b}{(n+2)^2}.$$

Analogously, we choose a = 1/3, b = 2/3, so that

$$u_{2,n} = \frac{4}{3\,(n+2)^2}.$$

Then

$$r_{3,n} = \frac{n^2}{(n+3)^2}$$

which suggests

$$r_{k,n} = \frac{n^2}{(n+k)^2}.$$

Sticking with

$$s_{k,n} = a\,n + b,$$

$$u_{k,n} = \frac{(1-a(2k-1))\,n^2 + (2k-ak^2-2bk)\,n + (1-b)k^2}{(n+k)^2}$$

which says

$$a = \frac{1}{2\,k-1}, \qquad b = \frac{3\,k-2}{2\,(2\,k-1)}$$

$$u_{k,n} = \frac{k^3}{2\,(2\,k-1)\,(n+k)^2}$$

and, sure enough,

$$r_{k+1,n} = \frac{n^2}{(n+k+1)^2}.$$

So finally,

$$s_{k,n} = \frac{2\,n+3\,k-2}{2\,(2\,k-1)}$$

and

$$\text{zeta } 2 = \frac{3\,k}{2\,(2\,k-1)} \mathop{\mathrm{R}}_{k\geq 1} \frac{k^3}{2\,(2\,k-1)\,(k+1)^2}$$

$$= \frac{3}{2\,k} \mathop{\mathrm{R}}_{k\geq 1} \frac{k}{(k+1/2)} \frac{1}{4}$$

with a convergence rate of 2 bits/term.

But this is just a special case of our $\arcsin^2$ formula with $y = 1/2$, so that

$$\zeta 2 = \frac{3}{2}\,\frac{8}{2} \arcsin^2 \frac{1}{2} = \frac{\pi^2}{6}.$$

(I am indebted to Dr. John W. Wrench, Jr. of the Naval Ship Research and Development Center for pointing out that this zeta 2 formula is derived as a Markov transformation in Section 156 of Knopp's Theory and Application of Infinite Series. Later, however, we will improve this to a 6 bits/term formula, which has a better chance of being undiscovered.)

This example has nicely illustrated the actual practice of the method:

Acceleration Recipe

1) Write the term ratio $r_{0,n}$

2) Guess a form for a splitting function $s_{0,n}$, cunningly including some undetermined coefficients. It probably should be asymptotic to $1/(1-r_{0,n})$.

3) Choose these coefficients to simplifly

$$u_{0,n} = 1 - s_{0,n} + s_{0,n+1}\, r_{0,n}$$

4) Write $r_{1,n} = \dfrac{u_{0,n+1}}{u_{0,n}}\, r_{0,n}$

5) Guess $r_{k,n}$ so that it agrees with $r_{0,n}$ and $r_{1,n}$.

6) Write $r_{k+1,n} = \dfrac{u_{k,n+1}}{u_{k,n}}\, r_{k,n}$ and determine the coefficients in $s_{k,n}$ to achieve agreement with $r_{k,n}$.

This all boils down to searching for rational solutions to the recurrence relations defining $u_{k,n}$ and $r_{k,n}$,

and there may be much to gain from exploring the literature of nonlinear difference equations, which I have as yet neglected.

Now let's try harmonic numbers.

$$h_p = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} = \underset{n \geq 1}{\overset{p}{R}} \frac{n}{n+1}$$

We have not equipped our transform for finite series--it would evidently produce a sum of two series of orphans, one from the left and one from the right. We can, however, use the transform to see what the one on the left will be.

If we again try

$$s_{1,n} = a\,n + b$$

with

$$r_{1,n} = \frac{n}{n+1}$$

we find

$$u_{1,n} = \frac{n - b + 1}{n + 1}$$

and

$$r_{2,n} = \frac{n}{n + 1}$$

independent of a and b! This means that we can extract an arbitrary series of orphans from the left without affecting the term ratio at all.
Other choices of splitting functions are equally unenlightening. So it wants an infinite series? We give it an infinite series.

$$h_p = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$$

$$- \left(\frac{1}{p+1} + \frac{1}{p+2} + \frac{1}{p+3} + \dots \right.$$

$$= \sum_{n \geq 1} \frac{1}{n} - \frac{1}{n+p} = \sum_{n \geq 1} \frac{p}{n (n + p)}$$

A slightly less dubious way to get this is to let m approach infinity in the cute identity

$$\sum_{n=1}^{m} \frac{p}{n (n + p)} = \sum_{n=1}^{p} \frac{m}{n (n + m)}$$

but thie seems to rely on p being an integer--a restriction which we will profitably waive in order to conquer the digamma (Psi) function while we are at this:

$$h_p = \sum_{n \geq 1} \frac{p}{n (n + p)} = \mathrm{Psi}(1 + p) + \mathrm{enigma}$$

where enigma is Euler's constant = .5772156649... and is usually spelled gamma. We have

$$r_{1,n} = \frac{n (n + p)}{(n + 1) (n + p + 1)} .$$

Using good old

$$s_{1,n} = a n + b,$$

we get

$$u_{1,n} = \frac{(1-a)\, n^2 + (p-2b-a+2)\, n + (1-b)(p+1)}{(n + 1)\, (n + p + 1)}$$

which suggests

$$a = 1 \quad \text{and} \quad b = \frac{p + 1}{2}$$

so that

$$u_{1,n} = - \frac{(p - 1)\, (p + 1)}{2\, (n + 1)\, (n + p + 1)}$$

and

$$r_{2,n} = \frac{n\, (n + p)}{(n + 2)\, (n + p + 2)} .$$

Let us leap to the conclusion that

$$r_{k,n} = \frac{n\, (n + p)}{(n + k)\, (n + p + k)} .$$

Then

$$u_{k,n} = \frac{(1-(2k-1)a)\, n^2 + (p-2kb-(k^2+jk-p)a+2k)\, n + (1-b)k(p+k)}{(n + 1)\, (n + p + 1)}$$

so that if

$$a = \frac{1}{2\, k - 1} \quad \text{and} \quad b = \frac{3\, k + p - 2}{4\, k - 2}$$

giving

$$s_{k,n} = \frac{2\, n + 3\, k + p - 2}{2\, (2\, k - 1)}$$

then

$$u_{k,n} = \frac{k\, (k + p)\, (k - p)}{2\, (2\, k - 1)\, (n + k)\, (n + k + p)}$$

and lo and behold

$$r_{k+1,n} = \frac{n\ (n + p)}{(n + k + 1)\ (n + p + k + 1)} .$$

Finally

$$h_p = \frac{p}{p + 1} \prod_{n \ge 1} \frac{n}{n + 1} \frac{n + p}{n + p + 1}$$

$$= \frac{p}{p + 1} \prod_{n \ge 1} r_{1,n} = \frac{p}{p + 1} s_{k,1} \prod_{k \ge 1} u_{k,1}$$

$$= \frac{p}{p + 1} \frac{3\ k + p}{2\ (2\ k - 1)} \prod_{k \ge 1} \frac{k\ (k + p)\ (k - p)}{2\ (2\ k - 1)\ (k + 1)\ (k + p + 1)}$$

$$= \frac{p\ (3\ k + p)}{2\ k\ (k + p)} \prod_{k \ge 1} \frac{k - p}{k + 1/2} \frac{1}{4}$$

$$= - \sum_{k \ge 1} \frac{(3\ k + p) \binom{k - p - 1}{k}}{k\ (k + p) \binom{2\ k}{k}} .$$

As a consistency check, we can divide through by p, then let p approach 0, yielding the zeta 2 equation we got earlier.

Note that if p is an integer > 0, the k-p in the numerator of the ratio will terminate this series on the jth term. We have our finite series back, only it converges faster!

For instance,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{761}{280} =$$

$$\frac{44}{9} - \frac{49}{15} + \frac{238}{165} - \frac{5}{12} + \frac{46}{585} - \frac{13}{1386} + \frac{29}{45045} - \frac{1}{51480}$$

Now if we are computing harmonic numbers as exact rationals, it is easier to add up the first series than the second. But if we want big ones, this becomes impractical. (h_{99} involves 83 digits.) If we resort to decimal approximations,

we can still sum the harmonic series using approximate reciprocals, but even this grows impractical as p gets to be a few hundred. There is a divergent series involving log p, Euler's constant, and Bernoulli numbers, but even with an abundance of digits of these, you cannot extract many more than 2 pi p/ln 10 digits. The formula we have just derived seems to solve our problem to the tune of 2 bits/term, but on closer inspection, we see that the power of p in the numerator of the term ratio is one greater than in the denominator, so that if p is large, the series will not start to converge until k reaches (p - 2)/5.

To get the most out of this formula, we should notice a degree of freedom in the series transform that we have not yet exploited--namely, that setting n to m in the $s_{k,n}$ and $u_{k,n}$ of

the transformed series corresponds to starting the original series at the mth term. (We must also remember to scale both series so that the mth term of the original series is of the desired size.) Thus we have

$$h_{m,p} = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \dots$$

$$- \left(\frac{1}{m+p} + \frac{1}{m+p+1} + \frac{1}{m+p+2} + \dots \right.$$

$$= h_{m+p-1} - h_{m-1}$$

$$= \sum_{n \ge m} \frac{1}{n} - \frac{1}{n+p} = \sum_{n \ge m} \frac{p}{n\,(n + p)}$$

$$= \frac{p\,(3\,k + 2\,m + p - 2)}{2\,m\,(m + p)\,(2\,k - 1)} \mathop{R}_{k\ge 1} \frac{k\,(k + p)\,(k - p)}{2\,(2\,k - 1)\,(k + m)\,(k + m + p)}$$

$$= \frac{p\,(3\,k + 2\,m + p - 2)}{2\,m\,(m + p)} \mathop{R}_{k\ge 1} \frac{k\,(k + p)\,(k - p)}{(k + 1/2)\,(k + m)\,(k + m + p)}\,\frac{1}{4}$$

$$= \sum_{k\ge 0} \frac{p\,(3\,k + 2\,m + p + 1)\binom{k+p}{k}\binom{k-p}{k}}{2\,m\,(m + p)\,(2\,k + 1)\binom{2k}{k}\binom{k+m}{k}\binom{k+m+p}{k}}$$

Now if m is about as large as p, we see that this new term ratio is about 1/8, so that we have an almost 1 digit/term method for summing the reciprocals between half a trillion and a trillion, for instance. But we can use this same trick for those between a quarter trillion and half a trillion, etc., so we really can accurately compute very large harmonic numbers subject to a cost factor of $\log_2 p$.

This example shows that once we successfully accelerate a series, we have a handy way to sum finite but large intervals of that series, since the starting value of the summation index is merely a parameter in the new series. In fact, it need not be an integer:

We know

$$\cot z = \frac{1}{z} + \sum_{k\ge 1} \frac{2\,z}{z^2 - k^2\,\mathrm{pi}^2}$$

then

$$\mathrm{pi}\; z \cot \mathrm{pi}\; z = -\,z\,h_{-z,2z} - 1$$

$$= -1 + \sum_{k\ge 0} \frac{(3\,k + 1)\binom{k+2z}{k}\binom{k-2z}{k}}{(2\,k + 1)\binom{k+z}{k}\binom{k-z}{k}\binom{2k}{k}},$$

$$= -1 + (3\,k - 2) \mathop{R}_{k\ge 1} \frac{k\,(k^2 - 4\,z^2)}{(4\,k + 2)\,(k^2 - z^2)}$$

a 2 bits/term cotangent formula.

Similarly for Catalan's constant:

$$\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$$

$$= 2 \left(\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \ldots \right.$$

$$\left. - \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right. \right.$$

$$= \lim_{p \to 0} \frac{h_{1/4,p}}{16\ p} - \frac{h_{1/2,p}}{4\ p}$$

$$= \frac{3}{2} \sum_{k \geq 0} \frac{1}{\binom{k + 1/4}{k}^2 \binom{2\ k}{k}} - \frac{pi^2}{8}$$

$$= \frac{3}{2} \underset{k \geq 0}{R} \frac{(k + 1)^3}{(k + 5/4)^2\ (4\ k + 2)} - \frac{pi^2}{8}$$

Later on, when we accelerate eta 2, we will get a single series formula for Catalan's constant.

The next example will emphasize a very important point. We recall that our original goal was to accelerate the particular series

$$\frac{1}{1\ (1 + p)} + \frac{1}{2\ (2 + p)} + \frac{1}{3\ (3 + p)} + \ldots$$

with

$$r_{1,n} = \frac{n\ (n + p)}{(n + 1)\ (n + p + 1)}$$

but in the process, we accelerated the more general form

$$r_{k,n} = \frac{n\ (n + p)}{(n + k)\ (n + p + k)} .$$

This means that if we wish to accelerate some other series, and its term ratio is expressible as

$$r_{a,n+b}$$

for some a and b, then we can use the transform we already found, summing from k = a instead of 1, and setting m (the initial value of n in the original series) to b+1 instead of 1. Thus if we wish to accelerate the apparently very different series

$$\frac{4}{pi} = 1^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\ \frac{1}{4}\right)^2 + \left(\frac{-1}{2}\ \frac{1}{4}\ \frac{3}{6}\right)^2 + \left(\frac{-1}{2}\ \frac{1}{4}\ \frac{3}{6}\ \frac{5}{8}\right)^2 \ldots$$

(Jolley's Summation of Series, formula #274)

we find the term ratio $= \left(\frac{2\ n - 1}{2\ n + 2}\right)^2$.

But this is just $r_{3/2,n-1/2}$ with p = 0:

$$\mathop{R}_{n\geq 0} \left(\frac{2\ n - 1}{2\ n + 2}\right)^2 = \mathop{R}_{n\geq -1/2} \frac{n^2}{(n + 3/2)^2}$$

$$= \frac{3\ k - 3}{2\ (2\ k - 1)} \mathop{R}_{k\geq \frac{3}{2}} \frac{k^3}{(2\ k - 1)^3}$$

$$= \frac{3\ (2\ k - 1)}{8} \mathop{R}_{k\geq 1} \frac{(2\ k + 1)^3}{32\ k^2\ (k + 1)}$$

so

$$\frac{4}{pi} = 12 \sum_{k \geq 1} \frac{(2k-1)\, k^2 \binom{2k}{k}^3}{2^{8k}},$$

a formula which has among its oddities the property of speeding up, rather than slowing down, to its eventual convergence rate of 2 bits/term. This is the only pi series I can recall whose terms (and partial sums) are all finite decimal fractions, being integers over powers of 2 (a fact obscured, in this case, by R notation).

Many thanks are due Dr. Richard Pavelle of Perception Technology Inc., who first suggested transforming the harmonic series.

Let us now return to

$$pi/4 = \arctan 1 = 1 - 1/3 + 1/5 - 1/7 + \ldots$$

to see how we can improve on Euler's transform. We have

$$r_{0,n} = - \frac{2n+1}{2n+3}$$

which approaches -1, suggesting a splitting function approximating 1/2. A very simple non constant such function is apparently

$$s_{0,n} = \frac{n+a}{2n+b}$$

with a and b chosen to simplify

$$u_{0,n} = \frac{4\,(b-2a+1)\, n^2 + (2b^2 + 10b - 4ab - 12a + 4)\, n + 3b^2 + 5b - 4ab - 6a}{(2n+3)\,(2n+b)\,(2n+b+2)}$$

Reducing the degree of the numerator as before, we let $b = 2a - 1$. Then

$$u_{0,n} = \frac{2\,((2a-2)\, n + 2a^2 - 2a - 1)}{(2n+3)\,(2n+2a-1)\,(2n+2a+1)}$$

Instead of simply taking the obvious choice of $a = 1$, let us proceed directly to

$$r_{1,n} = \frac{u_{0,n+1}}{u_{0,n}} r_{0,n} =$$

$$- \frac{2 n + 1}{2 n + 5} \frac{2 n + 2 a - 1}{2 n + 2 a + 3} \frac{(2 a - 2) n + 2 a^2 - 3}{(2 a - 2) n + 2 a^2 - 2 a - 1}$$

to see which choices of a allow at least one of the denominator terms to cancel a numerator term. For each pairing of a term from the denominator with one from the numerator, we scale them so that the coefficients of n are equal, then solve for the a which makes the whole terms equal. When using MACSYMA, the very powerful symbolic math program at MIT, one can perform this operation succinctly by invoking the RESULTANT function, whose existence and usefulness were revealed to me by Professor Hal Abelson. The resultant of the numerator and denominator of the above expression is

$$- 2^{16} a^2 (a - 1)^2 (a - 2)^2 (a - 3) (a + 1) (2 a - 1) (2 a - 3)$$

and in general, the resultant of two polynomials is a polynomial in their coefficients which is 0 when they have a common factor or their leading coefficients simultaneously vanieh. They are painful to compute, but will no doubt come back into popularity now that machines will do them for us.

The series in question is so easy to accelerate that any of these choices of a will lead to a ratio which is in turn easy to split, etc., and in this way we can generate dozens of pi formulas of varying convergence rate. Perhaps the most interesting, yet easily found formula lies along the path indicated by choosing a = 3/2. Then

$$r_{1,n} = - \frac{(2 n + 2) (2 n + 3)}{(2 n + 5) (2 n + 6)} .$$

The simplest guess of $r_{k,n}$ consistent

with k = 0 and 1 is apparently

$$r_{k,n} = - \frac{(2 n + k + 1) (2 n + k + 2)}{(2 n + 3 k + 2) (2 n + 3 k + 3)}$$

which is, in fact, supported by

$$s_{k,n} = \frac{2 n + 5 k + 3}{4 n + 6 k + 4}$$

(which can be found with some pain by determining a and b

in $s_{k,n} = \dfrac{n + a}{2 n + b}$ so that $r_{k+1,n}$ conforms to our conjectured

formula for $r_{k,n}$). Then

$$u_{k,n} = \frac{(k + 1)\ (2 k + 1)\ (2 n + k + 1)}{(2 n + 3 k + 2)\ (2 n + 3 k + 3)\ (2 n + 3 k + 4)}$$

so that

$$\frac{pi}{4} = \frac{5 k + 3}{4} \mathop{R}_{k\geq 0} \frac{(k + 1/2)\ (k + 1)}{(k + 4/3)\ (k + 5/3)} \frac{2}{27},$$

$$= \sum_{k\geq 0} \frac{(5 k + 3)\ (2 k)!\ k!}{2^{k+1}\ (3 k + 2)!}$$

at better than 1 decimal digit/term. Interestingly, this series can be simplified by rerunning the transform just one step on this new series, with

$$r_{0,k} = \frac{(5 k + 8)\ (k + 1/2)\ (k + 1)}{(5 k + 3)\ (k + 4/3)\ (k + 5/3)} \frac{2}{27}$$

and

$$s_{0,k} = \frac{6 k + 4}{5 k + 3}$$

which happens to be the reciprocal of the old $s_{k,0}$.

Then

$$u_{0,k} = - \frac{5 (k + 1)\ (k + 2)}{3\ (3 k + 4)\ (5 k + 3)}$$

so

$$r_{1,k} = \frac{(k + 1/2)\ (k + 3)}{(k + 5/3)\ (k + 7/3)}\ \frac{2}{27}$$

and thus

$$\frac{pi}{4} = 1 - \frac{5}{24} \mathop{\mathrm{R}}_{k \ge 0} \frac{(k + 1/2)\ (k + 3)}{(k + 5/3)\ (k + 7/3)}\ \frac{2}{27}$$

or

$$pi = 4 - \sum_{k \ge 0} \frac{5}{2^k\ (2\ k + 1)\ \binom{3\ k + 4}{k + 2}} .$$

But this formula is just what we would have gotten had we accelerated the series

$$\frac{pi}{4} - 1 = - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ . \ . \ .$$

and then restored the initial 1; which is to say that

$s_{k,1} \mathop{\mathrm{R}}_{k \ge 0} u_{k,1}$ can be simplified further than $s_{k,0} \mathop{\mathrm{R}}_{k \ge 0} u_{k,0}$.

This suggests that when we get an acceleration formula, we should check to see if there are any preferable starting values of the summation index which differ by an integer from the the one initially intended. The brute force way to do this is to rewrite

$$s_{k,m} \mathop{\mathrm{R}}_{k \ge 0} u_{k,m} \quad \text{as} \quad s_{0,m} \mathop{\mathrm{R}}_{k \ge 0} \frac{s_{k,m+1}}{s_{k,m}}\ u_{k,m}$$

and then take the resultant with respect to k of the numerator and denominator of the expression on the right to determine the values of m for which it reduces.

Actually, the fact that there were two ways to find that simplification in the foregoing pi/4 formula was no coincidence, but rather a manifestation of a beautiful duality between a series and its transform:

If $s_{k,n}$ splits the series whose ratio is $r_{k,n}$ to make the series

with ratio $r_{n,k} = \dfrac{s_{k+1,n}}{s_{k,n}} u_{k,n}$, then $s_{n,k} = 1/s_{k,n}$ splits

$r_{n,k}$ back to $r_{k,n}$. (Note the switching subscripts.)

This can be verified directly from the recurrences defining $r_{k,n}$ and $u_{k,n}$.

Thus that extra split by the reciprocal of $s_{k,0}$ that we performed

on the transformed series was equivalent to the orphaning of the first term of the original series via the dual transform.

Just to make sure the reader understands this duality, an example:

As we will later show,

$$s_{k,n} = \frac{n + 3 k + 1}{2 n + 4 k + 1}$$

splits

$$r_{k,n} = - \frac{2 n - 1}{2 n + 4 k + 1}$$

to make

$$u_{k,n} = \frac{2 k + 1}{2 n + 4 k + 1} \frac{2 k + 2}{2 n + 4 k + 3}$$

so that the transformed series will have ratio

$$r_{n,k} = \frac{3 k + n + 4}{3 k + n + 1} \frac{2 k + 1}{4 k + 2 n + 3} \frac{2 k + 2}{4 k + 2 n + 5} .$$

By duality then,

$$s_{k,n} = \frac{4 n + 2 k + 1}{3 n + k + 1}$$

splits

$$r_{k,n} = \frac{3 n + k + 4}{3 n + k + 1} \frac{2 n + 1}{4 n + 2 k + 3} \frac{2 n + 2}{4 n + 2 k + 5}$$

to make

$$u_{k,n} = - \frac{3 n + k + 2}{3 n + k + 1} \frac{2 k - 1}{4 n + 2 k + 3}$$

with the ratio

$$r_{n,k} = - \frac{2 k - 1}{2 k + 4 n + 1} .$$

Zeta 3

$$\zeta 3 = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots = \sum_{n \ge 1} \frac{1}{n^3}$$

$$= \frac{5}{2} \sum_{k \ge 1} \frac{(-)^{k-1}}{k^3 \binom{2k}{k}},$$

the improvement being asymptotically 2 bits/term versus 0 bits/term. To be honest, I first got this formula from a Kummer transform (omitted for brevity), so it is probably known. We have

$$r_{0,n} = \frac{n^3}{(n+1)^3}.$$

If we choose $s_{0,n} = a\,n + b$, we find that a must be 1/2 to make the degree of the numerator of $u_{0,n}$ less than the degree of the denominator. But then we will find that there is no value of b for which $r_{1,n}$ is again a ratio of cubics, and experience indicates that the larger the degree of the term ratio, the harder it is to find a splitting function which keeps the degree from increasing yet further. The next simplest splitting function asymptotic to n/2 is a quadratic divided by a linear polynomial, which will contain 3 coefficients to be determined. Now we could begin a massive search through all values of these coefficients for which $r_{1,n}$ simplifies, but first, let me suggest two heuristics useful for limiting such searches--

1) Choose the denominator of the splitting function s to have factors in common with the denominator of the term ratio.

2) Choose the numerator of 1 - s to have factors in common with the numerator of the term ratio.

While these heuristics are not foolproof, a glance at the formula defining $u_{k,n}$ will show why they tend to produce simple u's.

If we use these heuristics to eliminate all three bugger factors, we get

$$s_{0,n} = 1 + \frac{n^2}{2\ (n + 1)}$$

which rewards us with

$$r_{1,n} = \frac{n\ (n + 1)^2}{(n + 2)^2\ (n + 3)}$$

suggesting

$$r_{k,n} = \frac{n\ (n + k)^2}{(n + k + 1)^2\ (n + 2\ k + 1)} .$$

But there is no reason to believe that $s_{k,n}$ should be asymptotic to n/2 for all k. To find out what it should be, we temporarily set $s_{k,n}$ to a n and find that a should be 1/(2 k + 2) in order to reduce the degree of $u_{k,n}$. Now the factors of the numerator and denominator of r are not all the same, so which do we choose for s? We need another heuristic:

3) Choose the largest factors.

This dictates

$$s_{k,n} = 1 + \frac{(n + k)^2}{2\ (k + 1)\ (n + 2\ k + 1)}$$

which is indeed the winner among the eight candidates. Then

$$u_{k,n} = - \frac{(k + 1)^2 (n + k)^2}{(n + 2 k + 2) (n + k + 1)^2 (n + 2 k + 1)}$$

and since $s_{k,1} = 5/4$,

$$\text{zeta } 3 = \frac{5}{4 k^2} \underset{k\geq 1}{R} \frac{- k}{2 (2 k + 1)}$$

Which is the R way of saying

$$= \frac{5}{2} \sum_{k\geq 1} \frac{(-)^{k-1}}{k^3 \binom{2 k}{k}} .$$

Again we notice a similarity to the arcsin^2 formula, this time with $y = i/2$, but with with an extra factor of $1/k$ in each term, which we can explain by dividing by y and integrating:

$$\text{zeta } 3 = 10 \int_0^{\frac{1}{2}} \frac{(\text{asinh } y)^2}{y} dy$$

where asinh is inverse hyperbolic sine.
In an effort to rewrite this as the usual integral definition of zeta 3, I tried substituting

$$y = \sinh t/2 .$$

Use

$$\coth t/2 = 1 + \frac{2}{e^t - 1}$$

and

$$\int_0^{\infty} f(x)\ dx = \int_0^{\infty} f(x)\ dx - \int_0^{\infty} f(x+a)\ dx$$

and

$$\mathrm{zeta}(n;x) = \frac{1}{(n-1)!} \quad \frac{x\, t^{n-1}}{e^{t} - x}\, dt$$

where

$$\mathrm{zeta}(n;x) = \sum_{k\geq 1} \frac{x^{k}}{k^{n}},$$

the generating function of the -nth powers. ($\mathrm{zeta}(n;1) = \mathrm{zeta}\ n$ and $\mathrm{zeta}(1;x) = -\log(1-x)$). Then expanding the numerator into 3 terms and using

$$\mathrm{zeta}\left(2; \frac{1}{\Phi^{2}}\right) = \frac{\pi^{2}}{15} - (\ln \Phi)^{2}$$

(see J. Edwards' Treatise on Integral Calculus, p. 285+), one winds up with

$$\mathrm{zeta}\ 3 = \frac{5}{4}\, \mathrm{zeta}\left(3; \frac{1}{\Phi^{2}}\right) + \frac{\pi^{2}}{6} \log \Phi - \frac{5}{6} (\log \Phi)^{3}$$

where Φ is the golden ratio

$$= 1.618\ldots = \Phi^{2} - 1 = e^{\operatorname{asinh} 1/2}.$$

Surprisingly, this formula appears in problem XXVII 41 iii of Edwards, credited to the ingenious manipulations of Landen. Landen and Ramanujan independently found the analogous relation:

$$\mathrm{zeta}\ 3 = \frac{8}{7}\, \mathrm{zeta}\left(3; \frac{1}{2}\right) + \frac{2\pi^{2}}{84} \log 2 - \frac{1}{42} (\log 2)^{3},$$

which suggests that there is at least one more of these identities, with ϕ instead of ϕ^2 or 2. Perhaps this is what was garbled to produce the obviously false equation in problem 41 iv.

Clausen, in Crelle's Journal, vol. 5, (1830), p38, gives

$$\text{zeta } 3 = \frac{\text{pi}}{8\,n^2} \; R_{n\geq\frac{1}{2}} \; \frac{n}{2\,(2\,n+1)} - \frac{3}{8\,n^2} \; R_{n\geq 1} \; \frac{n}{2\,(2\,n+1)} .$$

Noting that these R's are integrals of (arcsin y)/y and $(\text{arcsin } y)^2/y$ respectively, we can perform the analogous manipulations (starting with y = sin t/2i) to get

$$\sum_{n\geq 0} \frac{(-)^n}{(3\,n+1)^3} = \frac{5\,\text{pi}^3}{2\;3^{9/2}} + \frac{13\text{ zeta }3}{36}$$

and

$$\sum_{n\geq 0} \frac{(-)^n}{(3\,n+2)^3} = \frac{5\,\text{pi}^3}{2\;3^{9/2}} - \frac{13\text{ zeta }3}{36}$$

by separate consideration of the real and imaginary parts.

Returning to the zeta 3 acceleration formula, we note that setting n to 1/4 instead of 1 gives us a formula for

$$\frac{1}{1^3} + \frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{13^3} + \ldots$$

$$= \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \ldots\right.$$

$$\left. + \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right)$$

$$= \frac{1}{2}\left(\mathrm{zeta}\ 3 - \frac{\mathrm{zeta}\ 3}{8} + \frac{\mathrm{pi}^3}{32}\right)$$

so

$$\frac{7\ \mathrm{zeta}\ 3}{16} + \frac{\mathrm{pi}^3}{64} = \frac{80\ k^2 + 112\ k + 41}{40\ (k+1)\ (4\ k+1)^2} \ \underset{k \geq 0}{\mathrm{R}} \ \frac{-16\ (k+1)^2}{(8\ k+9)\ (8\ k+13)}$$

Splitting Function Addition Formula

Sometimes it is desirable or even necessary to choose a sequence of splitting functions whose formulas alternate between two different forms. The most useful special case of this is when every other s is identically 1. This has the effect of orphaning the first term of each partially accelerated series, replacing n by n + 1 in the term ratio. As we shall see, this is useful for increasing convergence rates, but more important, it allows us to use acceleration formulas which would otherwise involve division by 0.

Consider again the problem of accelerating zeta 3, with

$$r_{0,n} = \frac{n^3}{(n+1)^3}.$$

Before finding better heuristics, I found, by analogy with the Kummer transform,

$$s_{0,n} = \frac{n^2}{2\,(n-1)}$$

giving

$$u_{0,n} = -\frac{1}{(n-1)\,(n+1)}$$

and

$$r_{1,n} = \frac{n^2\,(n-1)}{(n+1)^2\,(n+2)}$$

But $s_{0,1}$ and $u_{0,1}$ don't exist, a problem which we might sidestep by remembering to sum from n = 2 instead of 1, adding back in the initial term when we finish. Blithely proceeding, we will eventually discover the pair

$$r_{k,n} = \frac{n^2\,(n-k)}{(n+1)^2\,(n+k+1)}$$

$$s_{k,n} = \frac{n^2}{2\,(k+1)\,(n-k-1)}$$

with

$$u_{k,n} = - \frac{(k + 1)^2}{(n + k + 1)\ (n - k - 1)} .$$

But this says that if we start with n=2, we will get a zero denominator in the k=1 orphan. So we must translate the k=1 series by a term as we did for k = 0. In fact, we will have to do this for every k. A nice way to think of this is that before the application of the kth splitting function $s_{k,n}$, we interject the splitting function

$s_n = 1$. Thus, before the k=0 split, $r_{0,n}$ will become

$$\frac{(n + 1)^3}{(n + 2)^3}$$

necessitating the replacement of n by n + 1 in $s_{0,n}$, which

evades the division by 0. By extension, $r_{k,n}$ will become

$$\frac{n\ (n + k)^2}{(n + k + 1)^2\ (n + 2\ k + 1)}$$

just before the k + 1st replacement of n by n + 1, after which the value of $s_{k,n}$ must be

$$\frac{(n + k + 1)^2}{2\ (k + 1)\ n}$$

(i.e. n + k + 1 replaces n). We note that the r formula now corresponds to the one we got using the heuristics, but the s formula does not. This is because the s in the earlier acceleration combined the effects of the current s and the s = 1 preceeding it.

How could we have combined these two alternating splitting functions to get the earlier one, had we not already known it? "Luckily", some straightforward algebra solves the more general problem of finding the splitting function s_n equivalent to successively splitting by p_n and then q_n:

Splitting Function Addition Formula

$$s_n = p_n + q_n (1 - p_n + p_{n+1} r_n)$$

In our case, $p_n = 1$ and

$$q_n = s_{k,n} = \frac{(n + k + 1)^2}{2 (k + 1) n} .$$

Using

$$r_n = r_{k,n+k} = \frac{(n + k)^2 n}{(n + k + 1)^2 (n + 2 k + 1)} ,$$

our new, compound splitting function becomes

$$s_{k,n} = 1 + \frac{(n + k)^2}{2 (k + 1) (n + 2 k + 1)}$$

in accordance with the earlier transform.

The main point of all this is that once we successfully transform a series by finding an s-r pair, we can find any number of new s-r pairs by interspersing the unit splitting function using this important special case of the addition formula. We summarize it under the name

Translation Transform

$$s_{k,n} \leftarrow 1 + s_{k,n+k+1} \; r_{k,n+k}$$

$$r_{k,n} \leftarrow r_{k,n+k}$$

.

Rich Schroeppel first proposed the technique of periodically translating a series while accelerating it, originally for the purpose of improving the convergence rate. Let us use this on some accelerations to see how this improves on the improvement. First we derive another pi formula from arctan 1:

$$r_{0,n} = - \frac{2n - 1}{2n + 1}$$

(with n starting at 1). Heuristically choosing the same denominator for the splitting function, while requiring it to approach 1/2 for large n, we get the form

$$s_{0,n} = \frac{n + a}{2n + 1}$$

giving

$$u_{0,n} = \frac{2\,((2 - 2a)\,n + 2 - a)}{(2n + 1)\,(2n + 3)}$$

suggesting $a = 1$ so that

$$r_{1,n} = - \frac{2n - 1}{2n + 5}$$

suggesting

$$r_{k,n} = - \frac{2n - 1}{2n + 4k + 1}$$

requiring $a = 3k + 1$ (in the formula for $u_{k,n}$ not shown)

so that

$$s_{k,n} = \frac{n + 3k + 1}{2n + 4k + 1}$$

giving

$$u_{k,n} = \frac{(2k + 1)(2k + 2)}{(2n + 4k + 1)(2n + 4k + 3)} .$$

Then

$$\frac{pi}{4} = \frac{3k + 2}{3} \underset{k \geq 0}{R} \frac{(k + 1/2)(k + 1)}{(k + 5/4)(k + 7/4)} \frac{1}{4} ,$$

with a modest 2 bits/term. (The author is mildly embarrassed to later discover that this formula is merely the result of pairwise grouping the terms of Euler's transform of the arctan 1 series, and could have been computed directly by using $p_n = 1/2 = q_n$ in the addition formula.)

Using the Translation Transform, we get

$$r_{k,n} = - \frac{2n + 2k - 1}{2n + 6k + 1}$$

$$s_{k,n} = 1 - \frac{n + 4k + 2}{2n + 6k + 1} \frac{2n + 2k - 1}{2n + 6k + 3}$$

These give

$$\frac{pi}{4} = \frac{2(7k + 6)}{15} \underset{k \geq 0}{R} - \frac{k + 1/2}{k + 7/6} \frac{k + 1}{k + 11/6} \frac{1}{27}$$

with a healthy log 27 = 1.43 decimal digits/term. We could have gotten this pi formula directly by accelerating arcsin 1/2, for which

$$r_{k,n} = \frac{(2n + 1)(2n + 2k + 1)}{4(2n + 2k + 2)(2n + 6k + 3)} ,$$

$$s_{k,n} = 1 + \frac{(2n + 2k + 1)(2n + 2k + 3)}{3(2n + 6k + 3)(2n + 6k + 5)} ,$$

and

$$u_{k,n} = -\frac{2\,(2k+1)\,(k+1)^2\,(2n+2k+1)}{(n+k+1)\,(2n+6k+3)\,(2n+6k+5)\,(2n+6k+7)}$$

starting n and k at 0.

Recalling the fact that offsetting n by 1/2 changes arcsin y into $(\arcsin^2 y)/y$, we have

$$\frac{pi^2}{18} = \frac{7k+5}{9\,(2k+1)} \;\underset{k\geq 0}{R}\; \frac{-(k+1)^2}{3\,(3k+4)\,(3k+5)}$$

$$= \frac{2}{9} \sum_{k\geq 0} \frac{(-)^k\,(7k+5)\,k!^3}{(2k+1)\,(3k+2)!}$$

This last formula can also be found by a straightforward acceleration and translation transform of eta 2:

$$\text{eta } 2 = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{pi^2}{12}.$$

Here $r_{k,n} = -\dfrac{n\,(n+k)}{(n+k+1)\,(n+2k+1)}$

and $s_{k,n} = 1 - \dfrac{n+k}{2\,(n+2k+1)}$

(consistent with the heurisitics)

so $u_{k,n} = \dfrac{(k+1)^2\,(n+k)}{(n+k+1)\,(n+2k+1)\,(n+2k+2)}$

and $\text{eta } 2 = \dfrac{3}{4} \;\underset{k\geq 1}{R}\; \dfrac{k^2}{(2k+1)\,(2k+2)}$

(one half times Markov's zeta 2 formula).

Then, translation transforming:

$$r_{k,n} = - \frac{n + k}{n + 2k + 1} \; \frac{n + 2k}{n + 3k + 1}$$

$$s_{k,n} = 1 - \frac{(n + k)(n + 2k)(n + 4k + 3)}{2(n + 2k + 1)(n + 3k + 1)(n + 3k + 2)}$$

$$u_{k,n} = - \frac{(k + 1)^2 (n + k)(n + 2k)}{(n + 2k + 2)(n + 3k + 1)(n + 3k + 2)(n + 3k + 3)}$$

whereupon $s_{0,1} \mathop{R}_{k\geq 0} u_{k,1}$ gives us 3/2 times the earlier formula

for $pi^2/18$. Curiously, $s_{k,5}$ factors, so that

$$\frac{pi^2}{12} - \frac{1}{1} + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} = - \sum_{k\geq 1} \frac{(-)^k (k + 3)(7k + 12)(k + 1)!^3}{k^2 (2k + 3)(3k + 4)!} .$$

Repeating the translation transform on the pi/4 formula, we get a series with 6 bits/term, but these terms have an unfactorable quadratic in their numerators. Noting, however, that $r_{0,3/2}$ is the term ratio for ln 2, we get

$$\ln 2 = s_{k,3/2} \mathop{R}_{k\geq 0} u_{k,3/2}$$

$$= \frac{14k + 11}{16} \mathop{R}_{k\geq 0} \frac{(k + 1)(2k + 1)}{8(4k + 5)(4k + 7)}$$

$$= \sum_{k\geq 0} \frac{3(14k + 11)}{(4k + 3)(2k + 1)\binom{4k + 1}{2k} 4^{k+2}}$$

Translation transforming the zeta 2 series, we get

$$r_{k,n} = \frac{(n+k)^2}{(n+2k)^2}$$

$$s_{k,n} = 1 + \frac{(2n+5k)(n+k)^2}{2(2k-1)(n+2k)^2}$$

$$u_{k,n} = \frac{k^3 (n+k)^3}{2(2k-1)(n+2k)^2 (n+2k+1)^2}$$

Here we must use $n = 0$ and sum from $k = 1$ so that $r_{k,n}$ is initially that of zeta 2:

$$\text{zeta } 2 = \frac{21k-8}{8(2k-1)} \underset{k\geq 1}{R} \frac{k^3}{8(2k-1)(2k+1)^2}$$

$$= \frac{21k-8}{8} \underset{k\geq 1}{R} \frac{k^3}{8(2k+1)^3}$$

$$= \frac{1}{8} \sum_{k\geq 0} \frac{(21k+13)\, k!^6}{(2k+1)!^3}$$

providing 6 bits/term.

Translation transforming the 4/pi formula,

$$r_{k,n} = \frac{(2n + 2k - 1)^2}{(2n + 4k + 2)^2}$$

$$s_{k,n} = 1 + \frac{4n + 10k + 7}{8(k + 1)} \; \frac{(2n + 2k - 1)^2}{(2n + 4k + 2)^2}$$

$$u_{k,n} = \frac{(2k + 3)^2 (2n + 2k - 1)^2}{8(k + 1)(2n + 4k + 2)^2 (2n + 4k + 4)^2}$$

Noting that $s_{k,2}$ reduces to

$$\frac{42k + 47}{32k + 32}$$

we get

$$\frac{4}{pi} = 1 + \frac{1}{4} + \frac{42k + 5}{2048} \; \underset{k \ge 1}{R} \; \frac{(2k + 1)^3}{512(k + 1)^3}$$

$$= \frac{5}{4} + \sum_{k \ge 1} \frac{(42k + 5) \binom{2k - 1}{k}^3}{2^{12k}}$$

which, like its predecessor, is also a series of exact binary fractions, its kth term being an integer of about 6k bits shifted about 12k bits to the right of the binary point, for a net gain of 6 bits/term. Jack Holloway (of the M. I. T. Artificial Intelligence Lab) points out that this enables us to compute the millionth digit of 1/pi without computing the first half million digits.

Translation transforming the pi formula whose ratio approached 2/27 (where we had set out to beat Euler's transform), we have

$$r_{k,n} = -\frac{(2n+3k+1)(2n+3k+2)}{(2n+5k+2)(2n+5k+3)}$$

$$s_{k,n} = 1 - \frac{(2n+3k+1)(2n+3k+2)(2n+7k+5)}{2(2n+5k+2)(2n+5k+3)(2n+5k+4)}$$

$$u_{k,n} = \frac{-(k+1)}{2n+5k+2}\ \frac{2k+1}{2n+5k+3}\ \frac{2n+3k+1}{2n+5k+4}\ \frac{2n+3k+2}{2n+5k+5}\ \frac{2n+3k+3}{2n+5k+6}$$

which promises a term ratio approaching 54/3125. But this rapid convergence is marred by the fact that $s_{k,0}$ has an unfactorable cubic numerator. Fortunately, $s_{k,1}$ does not, so we can again pull the trick of transforming the series with its first term borrowed. Thus

$$\frac{pi}{4} = 1 - \frac{(11k+13)(17k+12)}{720} \mathop{R}_{k\ge 0} - \frac{1}{5}\ \frac{2k+1}{5k+7}\ \frac{3k+3}{5k+8}\ \frac{3k+4}{5k+9}\ \frac{3k+5}{5k+11}$$

or

$$pi = 4 - \sum_{k\ge 0} \frac{(11k+13)(17k+12)}{3(2k+1)(k+1)(2k+3)\binom{5k+6}{2k+3}(-2)^k}$$

Translation transforming the special cases of the hypergeometric function near the beginning of this paper, we get a few familiar identities and a few weirdos. Recalling

$$F(a,b;c;z) = \mathop{R}_{n\geq 0} \frac{n+a}{n+c} \, \frac{n+b}{n+1} \, z$$

For $F(a,1;c;z)$ we will have

$$r_{k,n} = \frac{n+k+a}{n+2k+c} \, z$$

$$s_{k,n} = 1 - \frac{n+k+a}{n+2k+c} \, \frac{z}{z-1}$$

Then

$$F(a,1;c;z) = \frac{(z-2)k+(c-a)z-c}{c\,(z-1)} \mathop{R}_{k\geq 0} \frac{k+a}{2k+c+1} \, \frac{k+c-a}{2k+c+2} \, \frac{z^2}{z-1} .$$

Taking the resultant with respect to k to find simplifying values of a and c, we find that the degree of the term ratio reduces in the cases $c = 2a-2$, $2a-1$, $2a$, $2a+1$, and $2a+2$, but only in the case $c = 2a$ is the result expressible as an ordinary hypergeometric function:

$$F(a,1;2a;z) = \frac{z-2}{2z-2} F\left(a,1;a+1/2;\frac{z^2}{4z-4}\right)$$

In the other cases, the only way to assure the n+1 in the denominator of the term ratio requires particular values of z:

$$F\left(a,1;\ 2a-2;\ \frac{2a-4}{a-3}\right) = \frac{F\left(a-2,2;\ a-1/2;\ \frac{(a-2)^2}{(a-3)(a-1)}\right)}{(a-1)^2}$$

$$F\left(a,1;\ 2a-1;\ \frac{2a-3}{a-2}\right) = \frac{F\left(a-1,2;\ a+1/2;\ \frac{(a-3/2)^2}{(a-2)(a-1)}\right)}{(a-1)\ (2\ a-1)}$$

$$F\left(a,1;\ 2a+1;\ \frac{2a-1}{a}\right) = \frac{F\left(a,2;\ a+\frac{3}{2};\ \frac{(a-1/2)^2}{a\ (a-1)}\right)}{(1-a)\ (2\ a-1)}$$

$$F\left(a,1;\ 2*a+2;\ \frac{2a}{a+1}\right) = \frac{F\left(a,2;\ a+\frac{3}{2};\ \frac{a^2}{(a-1)(a+1)}\right)}{(1-a)\ (a+1)}$$

It is a manipulative pain in the neck to confirm a fact which is obvious when you think about it:

> Translation transforming an s-r pair and its dual produces equivalent formulas.

This is because the orphans split off by the 1's are made by the s's in the dual, and vice versa.

For some reason, discoveries rarely proceed along the most direct line. It is only afterwards that one sees how he could have found his result much more easily. An example is the next three zeta 3 formulas, which are derived in seemingly the reverse of logical order. This presentation has two advantages over Gaussian sanitization, in that besides providing a datum on human problem solving, it shows off the power of the splitting function addition formula in two unusual applications.

$$r_{0,n} = \frac{n^3}{(n+1)^3}.$$

Employing the heuristics, we try

$$s_{0,n} = 1 + \frac{(n+b)\,n^3}{a\,(n+1)^3}.$$

a must then be 2 to reduce the degree of u, while b = 2 zeros the leading coefficient in an ugly quadratic factor of u's numerator. The result is

$$r_{1,n} = \frac{(n+1)^3\,(2\,n+5)}{(n+3)^3\,(2\,n+3)}.$$

Only slightly discouraged by the fact that the ratio is now quartic, we should obviously try

$$s_{1,n} = 1 + \frac{(n+b)\,(n+1)^3}{a\,(n+3)^3}.$$

But an exhaustive application of the FACTOR and RESULTANT functions of MACSYMA fails to turn up any promising leads. Since zeta 3 formulas are so hard to find, we desperately try

$$s_{1,n} = 1 + \frac{(n + b)\ (n + 1)^3}{a\ (n + 3)^2\ (2\ n + 3)} .$$

Again a must be 2 to reduce the degree of u, and amazingly enough, b = 4 gives

$$r_{2,n} = \frac{(n + 2)^2\ (n + 3)}{(n + 4)\ (n + 5)^2} ,$$

a simplification to something more like the initial ratio. This suggests the possibility of a sequsnce of ratios alternately cubic and quartic, generated by an alternating sequence of splitters. If we succeed, we can use the addition formula to combine pairs of consecutive splitters to produce the compound splitter which takes us from one cubic form directly to the next. Anticipating this, we relabel the indices so that the cubic ratios have consecutive k's, while the quartic ones are distinguished by a prime ('). Thus, this last ratio (a cubic) will be renamed $r_{1,n}$, while the previous quartic will be called $r'_{0,n}$.

Now we can guess that

$$r_{k,n} = \frac{(n + 2\ k)^2\ (n + 3\ k)}{(n + 3\ k + 1)\ (n + 4\ k + 1)^2}$$

and we will shortly find that

$$s_{k,n} = 1 + \frac{(n + 5\ k + 2)\ (n + 2\ k)^2\ (n + 3\ k)}{(4\ k + 2)\ (n + 3\ k + 1)\ (n + 4\ k + 1)^2}$$

gives

$$r'_{k,n} = \frac{(n + 2k + 1)^2 (n + 3k + 1)(2n + 6k + 5)}{(n + 3k + 3)(n + 4k + 3)^2 (2n + 6k + 3)}$$

and then

$$s'_{k,n} = 1 + \frac{(n + 5k + 4)(n + 2k + 1)^2 (n + 3k + 1)}{(2k + 2)(2n + 6k + 3)(n + 4k + 3)^2}$$

indeed confirms

$$r_{k+1,n} = \frac{(n + 2k + 2)^2 (n + 3k + 3)}{(n + 3k + 4)(n + 4k + 5)^2}.$$

Now we combine s and s' in the addition formula and get a staggering expression for the compound splitting function involving an unfactorable polynomial of degree six in both k and n. Proceeding to compute u anyway, we determine that each of these ungainly terms will contribute a generous twelve bits:

$$s_{k,1} = \frac{63k^2 + 70k + 19}{8(2k + 1)(3k + 2)} + \frac{(k + 1)(3k + 1)(63k + 50)}{192(3k + 2)(4k + 3)^2}$$

$$u_{k,1} = \frac{(3k + 1)(k + 1)^4}{16(3k + 4)(4k + 3)^2 (4k + 5)^2}$$

$$\text{zeta } 3 = \frac{24570k^4 + 64161k^3 + 62152k^2 + 26427k + 4154}{3456(2k + 1)(3k + 1)(3k + 2)} \mathop{R}_{k\geq 0} \frac{(k + 1)^4}{16(4k+5)^2 (4k+7)^2}$$

$$= \sum_{k\geq 0} \frac{(24570k^4 + 64161k^3 + 62152k^2 + 26427k + 4154)\, k!^4 (2k+1)!^2}{48(2k + 1)(3k + 1)(3k + 2)(4k + 3)!^2}$$

This combination of cumbersome expression and rapid convergence is reminiscent of too many applications of the translation transform. Noting that the translation transform can be inverted, we ask what acceleration formula can be translation transformed to get this monster.

Inverse Translation Transform

$$s_{k,n} \leftarrow \frac{s_{k,n-k-1} - 1}{r_{k,n-k-1}}$$

$$r_{k,n} \leftarrow r_{k,n-k}$$

These relations follow from substituting n-k for n and n-k-1 for n in the translation transform, then solving for $r_{k,n}$ and $s_{k,n}$.

In the case in question, $s_{k,n}$ is too large to exhibit, but the result of the inverse translation transform is a barely tractable

$$r_{k,n} = \frac{(n + k)^2 (n + 2 k)}{(n + 2 k + 1) (n + 3 k + 1)^2}$$

$$s_{k,n} = 1 + \frac{(n+k)^2 (n+2k) (2n^3 + (20k+12) n^2 + (68k^2 + 83k+25) n + 80k^3 + 150k^2 + 93k+19)}{4 (2 k + 1) (n + 2 k + 1) (n + 3 k + 1)^2 (n + 3 k + 2)^2}$$

then $$s_{k,1} = 1 + \frac{(k + 1) (40 k + 29)}{36 (3 k + 2)^2}$$

and $$u_{k,1} = \frac{(k + 1)^4 (2 k + 1)}{9 (2 k + 3) (3 k + 2)^2 (3 k + 4)^2}$$

so that

$$\mathrm{zeta}\ 3 = \frac{364 k^2 + 501 k + 173}{144 (2 k + 1)} \mathop{R}_{k \ge 0} \frac{(k + 1)^4}{9 (3 k + 4)^2 (3 k + 5)^2}$$

$$= \sum_{k \ge 0} \frac{(364 k^2 + 501 k + 173) k!^6}{36 (2 k + 1) (3 k + 2)!^2}$$

which is somewhat nicer and still provides nearly 3 digits/term. Reapplying the inverse transform,

$$r_{k,n} = \frac{n^2 (n + k)}{(n + k + 1) (n + 2 k + 1)^2}$$

$$s_{k,n} = \frac{(k + 1) (n + 3 k + 2)}{4 (n + 2 k + 1)^2} + \frac{n + 3 k + 1}{2 (2 k + 1)}$$

so that

$$\mathrm{zeta}\ 3 = \frac{30 k + 19}{16 (k + 1) (2 k + 1)} \mathop{R}_{k \ge 0} \frac{(k + 1)^2}{4 (2 k + 3)^2}$$

$$= \sum_{k \ge 1} \frac{30 k - 11}{4 (2 k - 1) k^3 \binom{2 k}{k}^2}$$

which, at 4 bits/term, is similar to the very first zeta 3 formula we got, but twice as fast. It is, however, quite different from and computationally superior to grouping terms of the slower series pairwise, which produce an unfactorable cubic numerator.

In general, grouping terms pairwise tends to double the degree of the term ratio while doubling the digits/term, so if you intend to evaluate an R expressed series sequentially, the doubled computational effort per term will countervale the greater convergence rate. Also, it greatly simplifies machine computation of these series when the integers comprising the numerators and denominators in the R expression do not exceed the computer's register size.

On the other hand, Rich Schroeppel has noted that n iterations of pairwise term grouping will, for large enough n, evaluate 2^n terms of a sum much more rapidly than sequential evaluation, assuming the Fast Fourier (n log n) multiplication algorithm.

Combining terms pairwise is neatly done in R notation:

$$\ldots (b_{2k} + r_{2k} (b_{2k+1} + r_{2k+1} (\ldots$$

$$= \ldots (b_{2k} + r_{2k} b_{2k+1} + r_{2k} r_{2k+1} (\ldots$$

or

$$b_k \mathop{R}_{k \geq 0} r_k = (b_{2k} + r_{2k} b_{2k+1}) \mathop{R}_{k \geq 0} r_{2k} r_{2k+1} .$$

Pitfalls

Inverse translation transforming arcsin 1/2,

$$r_{k,n} = \frac{2n+1}{8n+8} \frac{2n-2k+1}{2n+4k+3}$$

$$s_{k,n} = \frac{8}{3} \frac{n}{2n-2k-1} \frac{2n+1}{2n+4k+3}$$

gives

$$u_{k,n} = -\frac{2k+1}{2n-2k-1} \frac{4k+4}{2n+4k+3} \frac{4k+4}{2n+4k+5}$$

$$r_{n,k} = \frac{2k+1}{2k-2n+3} \frac{4k+4}{4k+2n+5} \frac{4k+4}{4k+2n+7}.$$

Noting that this simplifies for $n = 1$, and that arcsin 1/2 begins

$$\frac{1}{2} (1 + \frac{1}{24} (1 + \ldots$$

so that $a_{0,1}$ is 1/48,

$$\frac{pi}{6} - \frac{1}{2} = \frac{1}{30} \underset{k \geq 0}{R} \frac{4k+4}{4k+7} \frac{4k+4}{4k+9}.$$

But this is absurd because it says that pi > 3 + 1/5, since the R expression is > 1. What went wrong? Recall that in the derivation of the transform, the kth partial sum of the orphans comprising the new series differs from the original sum by the kth partially accelerated series

$$\sum_{n \geq 1} a_{k,n} = a_{k,1} \underset{n \geq 1}{R} r_{k,n} = a_{0,1} u_{0,1} u_{1,1} \ldots u_{k,1} \underset{n \geq 1}{R} r_{k,n}$$

which we assumed went to zero with increasing k.

Now as k approaches infinity, $r_{k,n}$ approaches

$$- \frac{2\ n + 1}{16\ (n + 1)},$$

which is the term ratio for

$$(1 + 1/8)^{-1/2}$$

by the binomial theorem, so

$$\underset{n \geq 0}{R}\ - \frac{2\ n + 1}{16\ (n + 1)} = 1 - \frac{1}{16}\ \underset{n \geq 1}{R}\ - \frac{2\ n + 1}{16\ (n + 1)} = \frac{2^{3/2}}{3}.$$

Thus the R expression in our error term approaches

$$16\ (\ 1 - \frac{2^{3/2}}{3})$$

while the u's which are its coefficient form a convergent infinite product! We can, in fact, evaluate this product as the discrepancy between the two sides of the false equation. To do this, we must find out what

$$\frac{1}{30}\ \underset{n \geq 0}{R}\ \frac{4\ n + 4}{4\ n + 7}\ \frac{4\ n + 4}{4\ n + 9}$$

really equals. Writing

$$r_{k,n} = \frac{4\ n}{4\ n + 4\ k + 3}\ \frac{4\ n + 4\ k}{4\ n + 4\ k + 5}$$

$$s_{k,n} = 1 + \frac{n + k}{k + 1}$$

Then

$$u_{k,n} = - \frac{n + k}{k + 1}\ \frac{4\ k + 3}{4\ n + 4\ k + 3}\ \frac{4\ k + 5}{4\ n + 4\ k + 5}$$

(whose product over k, we note with relief, "diverges to zero" for $n > -1$) and

$$r_{1,k} = - \frac{4\ k + 3}{4\ k + 7}\ \frac{4\ k + 5}{4\ k + 9} .$$

But this is just the sum

$$\frac{1}{2}\ (\frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots$$

with its terms grouped pairwise. Using

$$\frac{\arctan \frac{2^{1/2}\ x}{1 - x^2}}{2^{1/2}} - x = \frac{x^3}{3} - \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \dots$$

evaluated at 1, we get

$$\frac{1}{30}\ \underset{n \geq 0}{R}\ \frac{4\ n + 4}{4\ n + 7}\ \frac{4\ n + 4}{4\ n + 9} = \frac{1}{2}\ (\frac{pi}{2^{3/2}} - 1)$$

Finally, then,

$$\prod_{k \geq 0} \frac{2\ k + 1}{2\ k - 1}\ \frac{4\ k + 4}{4\ k + 5}\ \frac{4\ k + 4}{4\ k + 7} = - \frac{3\ 2^{1/2}\ pi}{8}$$

This relation can be verified by multiplying by

$$\frac{\sin pi/4}{pi/4} = \prod_{k \geq 0} \frac{4\ k + 3}{4\ k + 4}\ \frac{4\ k + 5}{4\ k + 4}$$

to leave

$$\prod_{k \geq 0} \frac{2\ k + 1}{2\ k - 1}\ \frac{4\ k + 3}{4\ k + 7}$$

whose kth partial product telescopes to

$$\frac{2\ k + 1}{- 1}\ \frac{3}{4\ k + 7}$$

which approaches -3/2. More generally, we have the

Corrected Series Transform

$$\mathop{R}_{n\geq m} r_{j,n} = s_{k,m} \mathop{R}_{k\geq j} u_{k,m} + \prod_{k\geq j} u_{k,m} \mathop{R}_{n\geq m} r_n$$

where r_n is the limit of $r_{k,n}$ as k grows infinite.

In our particular case, $\prod u_{k,m}$ converges for all m, giving

$$\prod_{k\geq j} \frac{2k+1}{2k-2m+1} \frac{4k+4}{4k+2m+3} \frac{4k+4}{4k+2m+5}$$

as a sum of two sums divided by a third sum. I wish I had time to further investigate this connection between sums and products.

Hoping for another product formula by dualizing the transform that started all this, we are very unpleasantly surprised.

$$r_{k,n} = \frac{2n+1}{2n-2k+3} \frac{4n+4}{4n+2k+5} \frac{4n+4}{4n+2k+7}$$

$$s_{k,n} = \frac{3}{8} \frac{2n-2k+1}{k} \frac{4n+2k+3}{2k+1}$$

$$u_{k,n} = \frac{2k+3}{8k} \frac{2k-2n-1}{2k+4n+5}.$$

The product of the $u_{k,n}$ "diverges to 0", leading to a contradiction

of the fact that the two series differ by a finite product.

The problem here is very serious: each individual split is invalid! The quadratic growth of $s_{k,n}$ is too fast for the input series,

which converges like a reciprocal quadratic for all k.

It was most unnerving and enlightening of Eugene Salamin of the Draper Laboratories to point out that any series can be converted to any other series in a single split, since s_{n+1} can

always be chosen to satisfy the recurrence

$$1 - s_n + r_n s_{n+1} = \text{anything} .$$

Since each of our series transforms involves applying an infinite number of splits, how can we be sure we aren't perpetrating the wildest absurdities? Our salvation lies in zippers. To see what actually happens to our series, imagine a zipper traveling along, splitting it in two. Right behind it, another zipper is zipping it back together, but offset so that one tooth (the orphan) is left hanging off the front and there is a corresponding extra tooth on one side between the zippers. If the series were finite, this extra tooth would become the orphan on the right when the zipper reached the end. In symbols, when the zippers reach the pth term,

$$\sum_{n\geq 0} a_n = a_0 s_0 + \sum_{n=0}^{p-1} (1 - s_n) a_n + s_{n+1} a_{n+1}$$

$$+ (1 - s_p) a_p + \sum_{n>p} a_n .$$

If the original series converges, then the last sum will vanish as p grows, but this isn't necesarily true of that extra tooth, $(1 - s_p) a_p$. Thus, besides worrying about the product over k of the $u_{k,n}$, we must make sure that the limit as p grows large of $(1 - s_{k,p}) a_{k,p}$ approaches something (preferably 0) which we can some over k. At last, then, the

Really Truly Correct Series Transform

$$\underset{n\geq m}{R}\, r_{j,n} = s_{k,m} \underset{k\geq j}{R}\, u_{k,m} + \prod_{k\geq j} u_{k,m} \underset{n\geq m}{R}\, r_n + \sum_{k\geq j} t_k$$

where $t_k = \operatorname{Lim}_n (1 - s_{k,n})\, a_{k,n}$ as n approaches infinity,

$r_n = \operatorname{Lim}_k r_{k,n}$ as k approaches infinity, and, of course,

$u_{k,n} = 1 - s_{k,n} + s_{k,n+1}\, r_{k,n}$. Ordinarily, $s_{k,n}$ would be chosen

so that the product and the t_k were 0.

Summing the Reciprocal Fibonacci Numbers

By an involved process, I found

$$r_{k,n} = \frac{(-)^{k} f_{n}}{f_{n+2k+1}}$$

$$s_{k,n} = \frac{f_{n+4k+2} + (-)^{k} f_{n+2k+1}}{(2 f_{2k} + f_{2k+1})\, f_{n+2k+1}}$$

$$u_{k,n} = \frac{(-)^{n+k+1} f_{2k+1} f_{2k+2}}{(2 f_{2k} + f_{2k+1})\, f_{n+2k+1} f_{n+2k+2}}$$

where f_0, f_1, f_2, ... are the fibonacci numbers 0, 1, 1, ...

so that

$$\sum_{n \ge 1} \frac{1}{f_n} = \frac{f_{4k+3} + (-)^{k} f_{2k+2}}{f_{2k+1} f_{2k+2}} \ \mathrm{R}_{k \ge 0} \ \frac{(-)^{k}}{2 f_{2k+2} + f_{2k+3}}$$

$$= \sum_{k \ge 0} \frac{(f_{4k+3} + (-)^{k} f_{2k+2})\,(-)^{\frac{k(k-1)}{2}}}{f_{2k+1} f_{2k+2}\, g_1 g_3 \ldots g_{2k-1}},$$

where $g_n = 2 f_{n-1} + f_n = \phi^{n} + (-\frac{1}{\phi})^{n}$, the nth Lucas number.

This is an extremely rapidly convergent series, yielding about $k^2/5$ digits for k terms.

Similar manipulations yield

$$\sum_{n\geq 1} \frac{1}{\sinh nx} =$$

$$\sum_{k\geq 0} \frac{\sinh (2k+2)x - \sinh (4k+3)x}{\sinh(2k+1)x \sinh(2k+2)x (2-2\cosh x)(2-2\cosh 3x)\ldots(2-2\cosh(2k+1)x)}$$

More generally , if

$$f_n = x^n + c\, y^n$$

so that

$$f_{n+1} = (x+y)\, f_n - x\, y\, f_{n-1}$$

then we have

$$r_{k,n} = \frac{x^k y^k f_n}{f_{n+2k+1}}$$

$$s_{k,n} = \frac{f_{n+4k+2} - (x\, y)^{k+1} f_{n+2k+1}}{(x^k - y^{k+1})(x^{k+1} - y^k) f_{n+2k+1}}$$

$$u_{k,n} = \frac{c\,(x\, y)^{n+k} (x^{2k+1} - y^{2k+1})(x^{2k+2} - y^{2k+2})}{(x^k - y^{k+1})(x^{k+1} - y^k) f_{n+2k+1} f_{n+2k+2}}$$

so that $$\sum_{n\ge m} \frac{1}{f_n} =$$

$$\frac{f_{m+4k+2} - (xy)^{k+1} f_{m+2k+1}}{(x-1)(1-y) f_m f_{m+1}} \quad \underset{k\ge 0}{\mathrm{R}} \quad \frac{c\,(xy)^{m+k} (x^{2k+1} - y^{2k+1})(x^{2k+2} - y^{2k+2})}{(x^{k+1} - y^{k+2})(x^{k+2} - y^{k+1}) f_{m+2k+2} f_{m+2k+3}}$$

again with the convergence rate of a theta function.

The author is perplexed by the fact that terms in the right side of this equation blow up when $x^j = y^{j+1}$ or $x^{j+1} = y^j$ for some nonnegative integer j, while the left side does not. In fact, it is hard to see how equality could hold when x is fixed at 2, for example, and y is nearly 1. Numerical experiments, however, indicate that the equation indeed holds, the large term being rapidly eroded by later ones. This indicates that when such an offending j exists and is positive, the above expression, with $\mathrm{R}_{k>j}$ instead of $\mathrm{R}_{k\ge 0}$, must be 0.

If we really do wish to sum $1/(x^n - 1)$, (a so called Lambert series when $x = 1/z$), we must resort to sneaky tricks:

$$\sum_{n\ge 1} \frac{2}{x^n - 1} = \sum_{n\ge 1} \frac{1}{(x^{1/4})^n - (-1)^n} + \frac{1}{(-x^{1/4})^n - (-1)^n} .$$

$(|x| > 1)$

This holds because, on the right, term 2 m + 1 cancels term 4 m + 2, leaving only the terms where n = 4 m. These latter two sums avoid the division by zero when fed to the transform.

In other cases we can avoid the problem by using the identity

$$\frac{\left(\frac{c\,y^n}{x^n}\right)^k}{x^n - c\,y^n} - c^k\left(\frac{y^k}{x^{k+1}}\right)^n = \frac{\left(\frac{c\,y^n}{x^n}\right)^{k+1}}{x^n - c\,y^n}$$

repeatedly to establish the Kummer transform

$$\sum_{n\geq 0} \frac{1}{x^n - c\,y^n} = \sum_{k\geq 0} \frac{1}{\left(\frac{1}{c}\right)^k - \frac{1}{x}\left(\frac{y}{c\,x}\right)^k}$$

$(|x| > |y|)$.

One Last Zeta 3 Formula

Knopp, Chapter VIII exercise 128a, points out a relation between zeta 3 and the series with ratio

$$\frac{n^3}{(n+3)^3}.$$

Tasting blood, we propose

$$r_{k,n} = \frac{n^3}{(n+2k+1)^3}$$

as a candidate ratio in yet another zeta 3 transform. After guessing the form

$$s_{k,n} = 1 + \frac{a\,n^4 + b\,n^3 + c\,n^2 + d\,n + e}{(n+2k+1)^3}$$

we find (with just a little help from MACSYMA)

$$a = \frac{1}{6k+2}, \quad b = \frac{2k+1}{3k+1}$$

$$c = \frac{(2k+1)^3}{2\,(3k+1)\,(3k+2)}$$

$$d = -\frac{(k+1)\,(2k+1)^3}{2\,(3k+1)\,(3k+2)}$$

$$e = \frac{(k+1)^2\,(2k+1)^3}{3\,(3k+1)\,(3k+2)}.$$

Despite these composite coefficients, the author has so far failed to find any nicely factorable expression or sum of two expressions for this splitting function. Putting this function in some intelligible form could shed much light on how and when to look for such things. As it stands, the gross disparity in elegance between the ratio and its splitter seems to suggest that we should seek a new formulation of the transform process. In any case, taking partial fractions with respect to n,

$$s_{k,n} = \frac{n + 4k + 1}{2\,(3k + 1)} + \frac{(2k + 1)^3\,(3n^2 + (15k + 9)\,n + 20k^2 + 25k + 8)}{6\,(n + 2\,k + 1)^3\,(3\,k + 1)\,(3\,k + 2)}.$$

Then

$$\text{zeta } 3 = \frac{56\,k^2 + 80\,k + 29}{24\,(2\,k + 1)^3\,(k + 1)} \mathop{R}_{k\geq 0} - \frac{(k + 1)^2}{3\,(3\,k + 4)\,(3\,k + 5)}.$$

In the previous exercise, Knopp indicates that

$$\sum_{n\geq 1} \frac{1}{n^3\,(n + 1)^3} = 10 - pi^2$$

This is 1/8 (= the first term) of what we would get by substituting k + 1/2 for k in the last R expression for zeta 3, since

$$r_{1/2,n} = \frac{n^3}{(n + 2)^3}.$$

Euler's Constant

In the American Mathematical Monthly, vol. 76 #3, Mar69 p273, I. Gerst and H. F. Sandham give

$$\text{gamma} = \frac{1}{2} - \frac{1}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6} - \frac{2}{7} + \frac{3}{8} - \ldots - \frac{3}{15} + \frac{4}{16} - \ldots$$

the numerator increasing by one each time the denominator reaches a power of 2. We can write this as the double sum

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} \ldots$$

$$+ \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} \ldots$$

$$+ \frac{1}{8} - \frac{1}{9} + \frac{1}{10} \ldots$$

$$\vdots$$

which is the series for $-\log 2$ started at the 2^nth term, summed for $n > 0$. Recall the application of the translation transform to

$$r_{k,n} = - \frac{2n - 1}{2n + 4k + 1}$$

(the "embarrassing" formula) to get pi/4, since for $k = 0$ this is the term ratio for arctan 1. Replacing n by n + 1/2, and redefining

$$r_{k,n} = - \frac{n}{n + 2k + 1},$$

we have, for $k = 0$, the ratio we want. Replacing n by n + 1/2 in the corresponding splitting function and u,

$$s_{k,n} = \frac{2n + 6k + 3}{4(n + 2k + 1)}$$

$$u_{k,n} = \frac{(2k + 1)(2k + 2)}{4(n + 2k + 1)(n + 2k + 2)}$$

which says

$$\sum_{n \ge m} \frac{(-)^n}{n} = \frac{2m + 6k + 3}{4m(m+1)} \mathop{R}_{k \ge 0} \frac{(2k+1)(2k+2)}{4(2k+m+2)(2k+m+3)}$$

$$= (m-1)! \sum_{k \ge 0} \frac{(2m + 6k + 3)(2k)!}{4^{k+1} (m + 2k + 1)!}$$

(for m even). Thus, to get Eulers constant, we must

sum the R expression with $m = 2^j$, for $j > 0$.

$$\text{gamma} = \sum_{j>0} \sum_{k \ge 0} \frac{2^{j+1} + 6k + 3}{\binom{2^j + 2k + 1}{2k} 2^{j+2k+2} (2^j - 1)}$$

This prospect grows
cheerful when we notice that after a few j, the R expressions will converge with great rapidity, returning $2j + 2$ bits/term. This is because in the term ratio, there are two m's in the denominator and none in the numerator. Thus b bits of gamma will require about $(b-j)/(2j+2)$ terms of the jth series. This says that the total number of terms of all series will be about

$$\frac{b}{2} + \frac{b-1}{4} + \frac{b-2}{6} \ldots \frac{0}{2b} \sim \frac{b}{2} (\ln b - 1)$$

Zeta 2 Revisited

$$r_{k,n} = \frac{n\ (n + k)}{(n + k + 1)^2}$$

$$s_{k,n} = \frac{n + 2\ k + 1}{k + 1}$$

gives

$$u_{k,n} = - \frac{(k + 1)\ (n + k)}{(n + k + 1)^2}$$

Now $s_{k,1} = 2$ and

$$u_{k,1} = - \frac{(k + 1)^2}{(k + 2)^2}$$

which is a strange way to prove zeta 2 = 2 eta 2 .

Translation transforming,

$$\text{zeta } 2 = \frac{10\ k - 3}{4\ k\ (2\ k - 1)} \mathop{\mathrm{R}}_{k\geq 1} - \frac{k}{2\ (2\ k + 1)} .$$

This series has similar convergence to our first (Markov) one

$$\text{zeta } 2 = \frac{3}{2\ k} \mathop{\mathrm{R}}_{k\geq 1} \frac{k}{2\ (2\ k + 1)}$$

except that the former one alternates.

The zeta 2 term ratio

$$\frac{n^2}{(n+1)^2}$$

is a special case of

$$\frac{n+p}{n+r}\;\frac{n+q}{n+s}$$

Choosing the splitting function $a\,n + b$ will produce a new ratio of

$$\frac{n+p}{n+r+1}\;\frac{n+q}{n+s+1}\;\frac{Q(a,b;n+1)}{Q(a,b;n)}$$

Where Q is a polynomial quadratic in n and linear in a and b. Now a and b can be chosen to make Q divisible by any two of the quantities n + p , n + q , n + r , or n + s , or they can be chosen to zero one or both leading coefficients of Q. Thus for our new ratio we may have any one of the quantities

$$\frac{n+p}{n+r+1}\;\frac{n+q}{n+s+1}, \quad \frac{n+p}{n+r+1}\;\frac{n+q+1}{n+s+1},$$

$$\frac{n+p}{n+r+1}\;\frac{n+q}{n+s}, \quad \frac{n+p}{n+r+1}\;\frac{n+q}{n+s-1},$$

$$\frac{n+p}{n+r}\;\frac{n+q}{n+s}, \text{ or } \quad \frac{n+p+1}{n+r+1}\;\frac{n+q+1}{n+s+1},$$

or any of the above with p and q swapped or r and s swapped. Also, from the addition formula, we learn that $p_n = 1$ and

$q_n = -1/r_n$ combine to produce 0, the identity split,

and thus the splitting function

$$-\frac{n+r-1}{n+p-1}\;\frac{n+s-1}{n+q-1}$$

will perform a reverse translation (the opposite of $s_n = 1$)

producing

$$\frac{n + p - 1}{n + r - 1} \; \frac{n + q - 1}{n + s - 1} .$$

By combining the above operations, we see that it is possible to independently change any of the four terms of our original term ratio by any integer. Thus, any series whose ratio is of this form, with p, q, r, and s all integers, must evaluate to

$$c\,\pi^2 + d$$

with c and d rational, since the splitting process only adds and multiplies by rational quantities. Among these rational quantities is 0/0, which figures into those splits which would result in cancellation to a linear term ratio. For example, to get

$$\frac{n + p}{n + r + 1} \; \frac{n + q}{n + s - 1}$$

we need the splitting function

$$\frac{s - 1 - r}{s - 1 - p} \; \frac{n + s - 1}{s - 1 - q} .$$

Thus, if both denominator terms exceed both numerator terms, there is no way to pass through any ratio where a denominator term is as large as a numerator term. In this

way we are prevented from proving that π^2 is rational, since it is easy to show that any series with ratio

$$\frac{n + p}{n + r}$$

telescopes to a rational sum when with r is rational and $r - p$ is an integer. As a corollary, we observe that we can achieve such a cancellation if a denominator term initially exceeds one in the numerator, thus any series with ratio

$$\frac{n + p}{n + r} \; \frac{n + q}{n + s}$$

sums to a rational if p is rational, $r - p$ is an integer, and $q - s$ is a nonnegative integer.

A particularly neat special case of the above is the transform:

$$r_{k,n} = \frac{n^2}{(n + c)\ (n + k)}$$

$$s_{k,n} = \frac{n + c - 1}{k + c - 1}$$

$$u_{k,n} = \frac{k^2}{(k + c - 1)\ (n + k)}$$

which says

$$\underset{n \geq m}{R} \frac{n^2}{(n + c)\ (n + j)} = \frac{m + c - 1}{j + c - 1} \underset{k \geq j}{R} \frac{k^2}{(k + c)\ (k + m)}$$

i.e.

$$(j + c - 1) \underset{n \geq m}{R} \frac{n^2}{(n + c)\ (n + j)}$$

is a symmetric function in c, j, and m.

With all of these relations, there must be a neat irrationality of pi^2 proof in here somewhere.

For those readers who have found the treatment so far a bit on the Heaviside, warning: we are approaching a singularity--we will now attempt to accelerate the "series"

$$e\,E_1(1) \text{ "=" } 0! - 1! + 2! - 3! + \ldots$$

gotten from iteratively integrating by parts $e^{1-t}\,dt/t$ from 1 to infinity.

We have

$$r_{1,n} = -n .$$

This is a special case of

$$r_{k,n} = \frac{n\,(n+c)\,(n+d_k+1)}{(n+a+k)\,(n+d_k)}\,z$$

with $d_1 = c = a$ and $z = -1$.

$$s_{k,n} = \frac{-1}{z\,(n+d_k)}$$

splits this to yield

$$u_{k,n} = \frac{(1+z\,(d_k+k+a-c))\,n + (k+a)\,(1+z\,d_k)}{(n+d_k)\,(n+a+k)\,z}$$

which formally satisfies the recurrences if

$$d_{k+1} = \frac{(k+a)\,(1+z\,d_k)}{1+z\,(d_k+k+a-c)} .$$

This says that

$$\mathop{R}_{n\geq m} r_{j,n} \quad \text{"="} \quad \frac{-1}{z\,(m + d_j)} \mathop{R}_{k\geq j} v_{k,m}$$

where

$$v_{k,n} = \frac{1 + z\,(d_k + k + a - c)}{(n + a + k)\, z} =$$

$$\frac{1 + z\,(2\,k + 2\,a - c - 1)}{(n + a + k)\, z} - \frac{(k + a - 1)\,(k + a - c - 1)}{(n + a + k - 1)\,(n + a + k)\, v_{k-1}}$$

(using the recurrence relation for d_k to eliminate it).

Expressed in standard notation,

$$\sum_{n\geq m} a_{j,n} \quad \text{"="} \quad a_{j,m} \sum_{k\geq j} b_{k,m}$$

with $r_{k,n} = \dfrac{a_{k,n+1}}{a_{k,n}}$ and $v_{k,n} = \dfrac{b_{k+1,n}}{b_{k,n}}$

so that $b_{k+1,m} =$

$$\frac{1 + z\,(2\,k + 2\,a - c - 1)}{(m + a + k)\, z}\, b_{k,m} - \frac{k + a - 1}{m + a + k}\;\frac{k + a - c - 1}{m + a + k - 1}\; b_{k-1,m}$$

with $b_{j,m} = s_{j,m}$ and $b_{j+1,m} = s_{j,m}\, v_{j,m}$.

In the special case of $0! - 1! + 2! - \ldots$,
we have $j = m = 1$, $c = a = d_1$ = arbitrary, and $z = -1$.

Choosing c = 1, we have

$$\sum_{n\geq 0} (-)^n n! \text{ "=" } \sum_{k\geq 1} b_k$$

with $b_1 = 1/2$, $b_2 = 1/6$, and

$$b_{k+1} = \frac{2k-1}{k+2} b_k - \frac{k-1}{k+1} \frac{k}{k+2} b_{k-1} .$$

For reasons unknown to the author, this last series seems to oscillate with ever decreasing frequency and amplitude about the "correct" answer .596347..., gotten from the integral. The convergence is slow: summing through b_{76} gives

.599299... and through b_{100} gives .5935... .

Larger values of c tsnd to reducs the frequency of early oscillations and the amplitude of latsr ones, slightly improving the long term convergence.

Numerical Experiments

The reason that we choose $s_{k,n}$ approximately $= 1/(1 - r_{k,n})$ and not exactly is that $r_{k,n}$ usually becomes an explosively complicated expression after a few k. But there is nothing preventing us from numerically grinding out a few orphans to see how fast the series would go, could we find its formula. We try good old

$$pi = 4\ (1 - 1/3 + 1/5 - 1/7 + \ldots$$

and get the following data. (The orphans are consecutively numbered, with the cumulative sum on the next line. The correct portion of each sum is underlined. Sums and orphans have all been scaled up by a factor of 10^{69} and truncated to integers.)

```
0 3000000000000000000000000000000000000000000000000000000000000000000000
  3000000000000000000000000000000000000000000000000000000000000000000000
1  138888888888888888888888888888888888888888888888888888888888888888888
  3138888888888888888888888888888888888888888888888888888888888888888888
2    2672161849777295945438223644774843762995849932170626009194967146490
  3141561050738666184834327112533663732651884738821059514898083856035378
3          31352227689441396637992558518320650334026578222228694221244785286
  3141592402966355626230965105092182053302218765399281743592305100820664
4              249396611247391121684367925111189166634236262217058865571729697
  3141592652362966873622086789460107164491385399635543960651170672550361
5                  1231734176348781444565983771714828110936413086526108111308129
  3141592653594701049970868234026090936206213510571957047177278783858490
6                 - 7800184639717679772350514693761088428881202445735635954831
  3141592653586900865331150554253740421512452422143075844731543147903659
7                    2891758706224834123600203816344000323984340648544949096724
  3141592653589792624037375388377340625328796422467060185380088097000383
8                          920785467783247360695814560164132363740064579366477048
  3141592653589793544822843171624701321143356586599423925444667463477431
9                     - 307978293062687771788265106518483534771642225750468795
  3141592653589793236844550108936929532878250068115889153802441713008636
10                          1767408805137966711559705978848491072884617127544505
  3141592653589793238611958914074896244437956046964380226687058840553141
11                         - 241276143030542884996759163025924061191175058234964
  3141592653589793238370682771044353359441196883938456165495883782318177
12                             91705799864633482047373735623914069796610114571307
  3141592653589793238462388570908986841488570619562370235292493896889484
13                                21817337880486815568912123128461308427248586867
  314159265358979323846241038829686732830413953168549869660921145476351
14                                   - 25843147094975436508273013732023811464818370
  3141592653589793238462410362453720233328703023412484964577109680657981
15                                  232980224886899277781290877864536868277459031521
  3141592653589793238462643342678607132606484314290349501445387139689502
```

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16                                   30117601978615258825723607134583639645551667
 3141592653589793238462643372796209111221743140013956636829026785241169
17                                     843452815262727517168061497934656142244085
 3141592653589793238462643373639661926484470657182018133963682927485174
18                                 - 47883177665521571424556496866729050605936
 3141592653589793238462643373591778748818949085757461637096953876879268
19                                    9688659760934822448413038214658838658783602
 3141592653589793238462643383280438509753771534170439851755792535662870
20                                      - 221213928865692447844914858845300630
 3141592653589793238462643383280438288539842668478052006840933690362190
21                                        - 1808993349637380331178873274388527
 3141592653589793238462643383280438286730849318840671675662060415973663
22                                               337363528495513808425916164796
 3141592653589793238462643383280438286731186682369167189470486332138459
23                                 - 935556307196828079974140488069782438074
 31415926535897932384626433832795027304239898542891
24                                             260780474970340438891581727267086
 31415926535897932384626433832795027306847703292595
25                                                776379841139153325633620970538 6
 31415926535897932384626433832795027306925341276709
26                                               - 25200546405251219152236588 04
 31415926535897932384626433832795027306925316076162
27                                          226172366098368594081506560540838915
 31415926535897932384626433832795029568648977059848
28                                          - 14919586192479125137900359 35615
 31415926535897932384626433832795029568634057473656
29                                                    3063309693607465786726658 7
 31415926535897932384626433832795029568634057779987
30                                     - 72673167398180031715041434344190950
 314159265358979323846264338327950288419
31                                          - 16386665699367779252120813 90108
 31415926535897932384626433832795028841
32                                            98162261775859098713831934962 75
 314159265358979323846264338327950288419
33                                                       2709588918693310042830 8
 314159265358979323846264338327950288419
34                                                        222600222267351759834 1
 314159265358979323846264338327950288419
35                                                                 730531524281 7
 314159265358979323846264338327950288419
36                                          - 12465868455732609563211328510 58
 3141592653589793238462643383279502884197169 3
37                                                         - 2440670241739519 3
 3141592653589793238462643383279502884197169 3
```

We note several things:

1) The first six terms are positive, after which the signs are apparently random.

2) The series converges faster and faster until the first negative term, which provides only one additional correct digit.

3) After this, the terms grow erratically smaller until term 15, which has nearly 1000 times the magnitude of term 14 and yet provides four new correct digits.

4) Terms 18, 22, 26, 29, 31, 33, 34, 35, and 37 have the wrong sign, i.e. they actually make the sum worse. Term 31 actually undoes a good digit.

5) Term 36 is seventeen orders of magnitude larger than term 35 and provides five new digits, while the sizable term 27 overcompensates and contributes less than one.

6) The general trend after term 6 seems to be ever slower convergence.

Little wonder that we couldn't find a formula--I dare you to prove it converges at all!

I am indebted to Richard P. Howell of the MIT RLE PDP-1 for confirming that these results are not merely the PDP-10 bignum routines gone haywire.

Now in the early 1950's, Daniel Shanks did a Ph. D. thesis (Journal of Mathematics and Physics, 1955, first article) involving primarily numeric methods of sequence extrapolation.

Among them was a transform which he denoted $\tilde{e}_1$,

which, in effect, replaces each but the first term with the limit of the exponential sequence defined by that term's predecessor, itself, and its successor--the first member of the kth sequence produced by this process becoming the kth member of the transformed sequence. On page 5 Shanks gives a numerical example for which he too chose the 4 arctan 1 series (actually the sequence of partial sums). He computed only six transform terms to eight decimal places, since he was technologically constrained to electro-mechanical computation. Rich Schroeppel made the remarkable observation that if Shanks had started his sequence with a zeroth partial sum of 0, his transformed sequence would have been identical to the screwy sequence of partial sums that we got from splitting by 1/(1-r), and indeed, extending the computation of his example (without the initial 0) exposes the same erratic convergence. Although it is doubtful that he anticipated this, Shanks provides some valuable insight into this phenomenon with his section on spectra of sequences.

Miscellaneous Accelerations

In exercise 129c of chapter VIII, Knopp gives a relation completely equivalent to one step of a transform for eta 2 and Catalan's constant, but he seems to miss the significance of it, presumably because he was more interested in identifying series than in evaluating them. Rephrased in terms of splitting functions,

$$r_{k,n} = -\frac{n^2}{(n+2k)^2}$$

$$s_{k,n} = \frac{1}{2} + \frac{k}{2}\,\frac{2n+6k+1}{(n+2k)^2}$$

$$u_{k,n} = -\frac{k\,(2k+1)^3}{2\,(n+2k)^2\,(n+2k+1)^2}$$

Inverse translation transforming the "embarrassing" pi/4 formula:

$$r_{k,n} = -\frac{n-k}{n+3k+1}$$

$$s_{k,n} = \frac{1}{2} + \frac{(4k+1)\,(4k+5)}{16\,(n+3k+1)} - \frac{(4k+1)\,(2k+1)}{8\,(n+3k+2)} + \frac{4k+1}{16\,(n-k-1)}$$

$$u_{k,n} = -\frac{1}{2}\,\frac{k+1}{n-k-1}\,\frac{2k+1}{n+3k+1}\,\frac{4k+1}{n+3k+2}\,\frac{4k+3}{n+3k+3}$$

so if n starts at 1/2 and k starts at 0,

$$\frac{pi}{4} = \frac{11k+8}{15} \mathop{R}_{k\geq 0} \frac{8}{3}\,\frac{k+1}{2k+1}\,\frac{4k+1}{6k+7}\,\frac{4k+3}{6k+11}$$

$$= \sum_{k\geq 0} \frac{(11k+8)\,16^k\,k!\,(3k)!\,(4k)!}{3\,(6k+5)\,(2k)!\,(6k+1)!}$$

which, with a term ratio approaching 16/27, has the least finite convergence rate of any (rational) pi series I know.

Even worse is the 0 bits/term one gets by purposely choosing splitting functions which don't make u tend to zero with n:

$$r_{k,n} = -\frac{n+k}{n+k+2}\,\frac{n-k+2}{n-k+1}$$

$$s_{k,n} = 1 - \frac{2}{4k+3}\,\frac{n+k}{n-k+1}$$

$$u_{k,n} = -\frac{4k-1}{4k+3}\,\frac{n+k}{n+k+2}\,\frac{n-k}{n-k+1}.$$

Choosing n = -1/2, k = 0 (i.e. starting the arctan 1 series one term too early),

$$1 + \frac{pi}{4} = \frac{4k+5}{4k+3} \underset{k\geq 0}{R} - \frac{2k+1}{2k+3}\,\frac{4k-1}{4k+3}$$

$$= -\sum_{k\geq 0} \frac{(4k+5)\,(-1)^k}{(2k+1)\,(4k-1)\,(4k+3)}$$

which is no better than if we had taken terms of the original series pairwise. Grouping pairwise, however, would have destroyed the alternating sign. Instead, this series arises directly from arctan 1,

$$r_k = -\frac{2k+1}{2k+3},$$

with the single application of the splitting function

$$s_k = \frac{2k+1}{4k-1}.$$

Retransforming Euler's transform of arctan 1,

$$r_{0,n} = \frac{n}{2n+1},$$

I found

$$r_{k,n} = \frac{n+k}{2n+4k+3}\;\frac{2n+4k^2+6k+3}{2n+4k^2+6k+1}$$

$$s_{k,n} = \frac{4n+6k+3}{2n+4k+3}$$

$$u_{k,n} = -\frac{2k-1}{2n+4k+3}\;\frac{n+k}{2n+4k+5}\;\frac{2n+4k^2+14k+11}{2n+4k^2+6k+1}$$

which is very unlike the other accelerations in this paper due to the quadratic occurrence of k in r and u.

Translation transforming the foregoing, then initializing n to 1 and k to 0,

$$\frac{pi}{2} = \frac{8\,(14k^2+33k+20)}{105} \underset{k\ge 0}{R} -\frac{2}{3}\;\frac{k+2}{6k+11}\;\frac{2k-1}{6k+13}$$

Noting the similarity between this formula and our earlier

$$\frac{pi}{4} = \frac{2\,(7k+6)}{15} \underset{k\ge 0}{R} -\frac{k+1/2}{k+7/6}\;\frac{k+1}{k+11/6}\;\frac{1}{27},$$

we suspect a single application of some splitting function relates them. To find it we divide the first formula by 2 and rewrite the second formula so that they both have the same expression to the right of the R.

$$\frac{2\,(7k+6)}{15} \underset{k\ge 0}{R} -\frac{2}{3}\;\frac{k+1}{6k+7}\;\frac{2k+1}{6k+11}\;\frac{k+2}{k+2}\;\frac{6k+13}{6k+13}\;\frac{2k-1}{2k-1}$$

$$= \frac{2\,(7k+6)}{15}\;\frac{2k-1}{-1}\;\frac{6k+7}{7}\;\frac{1}{k+1} \underset{k\ge 0}{R} -\frac{2}{3}\;\frac{k+2}{6k+11}\;\frac{2k-1}{6k+13}$$

(Thrice using the rule $\mathop{R}\limits_{k\geq 0} \frac{b_{k+1}}{b_k} r_k = \frac{b_k}{b_0} \mathop{R}\limits_{k\geq 0} r_k$.)

Now we have two series of the form

$$b_k \mathop{R}\limits_{k\geq 0} r_k \quad \text{and} \quad c_k \mathop{R}\limits_{k\geq 0} r_k .$$

Performing a modified split on the first of these,

$$(b_k - t_k + t_k) \mathop{R}\limits_{k\geq 0} r_k = t_k \mathop{R}\limits_{k\geq 0} r_k + (b_k - t_k) \mathop{R}\limits_{k\geq 0} r_k$$

$$= t_0 + r_0 t_{k+1} \mathop{R}\limits_{k\geq 0} r_{k+1} + (b_k - t_k) \mathop{R}\limits_{k\geq 0} r_k$$

$$= t_0 + r_k t_{k+1} \mathop{R}\limits_{k\geq 0} r_k + (b_k - t_k) \mathop{R}\limits_{k\geq 0} r_k$$

$$= t_0 + (b_k - t_k + r_k t_{k+1}) \mathop{R}\limits_{k\geq 0} r_k$$

which we must make equal to $c_k \mathop{R}\limits_{k\geq 0} r_k$ with the right choice of t_k.

This means $t_0 = 0$ and

$$t_{k+1} = \frac{t_k - b_k + c_k}{r_k} .$$

In this case,

$$b_k = \frac{4 (14 k^2 + 33 k + 20)}{105}$$

$$c_k = - \frac{2 (7 k + 6) (2 k - 1) (6 k + 7)}{105 (k + 1)}$$

$$r_k = - \frac{2}{3} \frac{k + 2}{6 k + 11} \frac{2 k - 1}{6 k + 13} .$$

Factoring the first few numerical values of t_k, we can easily guess

$$t_k = \frac{2\,k}{k+1}\;\frac{6\,k+5}{5}\;\frac{6\,k+7}{7}$$

which is confimed by direct substitution into the recurrence. Noting

$$(b_k - t_k + r_k\, t_{k+1}) \mathop{\mathrm{R}}_{k\geq 0} r_k =$$

$$b_0 \left(1 - \frac{t_k}{b_k} + \frac{t_{k+1}\; r_k\; b_{k+1}}{b_{k+1}\; b_k}\right) \mathop{\mathrm{R}}_{k\geq 0} \frac{b_{k+1}}{b_k}\, r_k$$

we see that t_k/b_k is the ordinary splitting function

for the series with ratio $b_{k+1}\; r_k / b_k$.

Thus

$$s_k = \frac{3\,k\,(6\,k+5)\,(6\,k+7)}{2\,(k+1)\,(14\,k^2 + 33\,k + 20)}$$

splits

$$\frac{pi}{4} = \frac{4\,(14\,k^2 + 33\,k + 20)}{105} \mathop{\mathrm{R}}_{k\geq 0} -\frac{2}{3}\;\frac{k+2}{6\,k+11}\;\frac{2\,k-1}{6\,k+13}$$

to get

$$pi = \frac{2\,(7\,k+6)}{15} \mathop{\mathrm{R}}_{k\geq 0} -\frac{2}{3}\;\frac{k+1}{6\,k+7}\;\frac{2\,k+1}{6\,k+11}$$

eliminating that ugly quadratic. Perhaps these bothersome polynomials which arise so often in our faster accelerations are similarly dispensable. Perhaps there is even a way to do this to series whose convergence rate has been doubled several times by pairwise term grouping.

Negating both sides of the recurrence for t_k,

$$(-t_{k+1}) = \frac{(-t_k) - c_k + b_k}{r_k}$$

which says that if t_k takes b_k to c_k, then $-t_k$ takes c_k back to b_k.

Thus

$$s_k = \frac{3k}{7k+6} \frac{6k+5}{2k-1}$$

carries the second series back to the first.

Another example of alternating splitting functions:
First a preliminary split of

$$r_n = \frac{n}{2n+1}$$

by

$$s_n = \frac{4n+4}{2n+3}$$

to get

$$r_{0,n} = \frac{n}{2n+7} \frac{2n-3}{2n-5}.$$

Then

$$r_{k,n} = \frac{n+k}{2n-4k-5} \frac{2n-4k-3}{2n+8k+7}$$

is split by

$$p_{k,n} = \frac{4n+4k+4}{2n+8k+7}$$

to get

$$r'_{k,n} = \frac{n + k}{2n - 4k - 3} \; \frac{2n - 4k - 1}{2n + 8k + 11}$$

which is in turn split by

$$q_{k,n} = \frac{4n + 10k + 11}{2n + 8k + 11}$$

to get r_{k+1}. p and q combine in the addition formula to

$$s_{k,n} = p_{k,n} + q_{k,n} \; \frac{2n - 4k - 3}{2n - 4k - 5} \; \frac{6k + 5}{2n + 8k + 7} \; \frac{6k + 9}{2n + 8k + 9}$$

which gives a $u_{k,n}$ containing five linear terms over five linear terms

which approaches -27/512. $s_{k,n}$ is quartic, and for at least the small

poeitive integers, won't factor. The resulting pi formulas are dull. Mildly interesting is the fact that translation transforming yields a formula both faster and slightly simpler, but not enough to make it a good pi formula. (Unfactorable cubic on the left of R, four linear factors over four linear factors on the right, ratio approaches -27/3125.)

Somewhat different pi formulas result if the preliminary splitting function s_n is merely 2, which gives

$$r_{0,n} = \frac{n}{2n + 3} = r'_{-1,n+1}.$$

so that $q_{k-1,n+1}$ and $p_{k,n+1}$ can be combined with the addition

formula to get a different splitting function.

In general, the arguments to the addition formula my be interchanged by making the first argument a preliminary split on the first iteration. Unless one of them is identically 1 (the translation transform), the resulting formulas will differ significantly, but converge at the same rate.